Calculus Review

1) A Bunsen burner is used to heat water in a beaker. The data in the table shows the temperature every 5 seconds. The temperature of the water increased by 80° Celsius in 80 seconds.

Time, t (seconds)	Temperature, T ($^\circ C$)
0	20.0
5	40.0
10	53.3
15	62.9
20	70.0
25	75.6
30	80.0
35	83.6
40	86.7
45	89.2
50	91.4
55	93.3
60	95
65	96.5
70	97.8
75	98.9
80	100

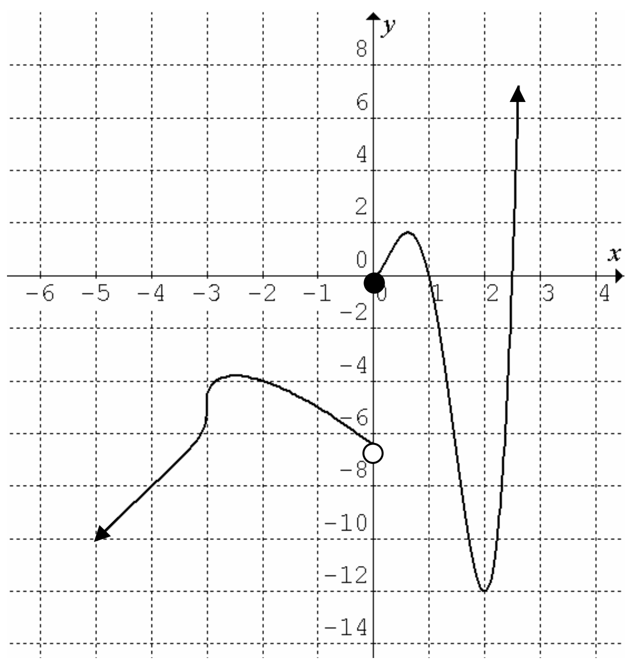
a) Find the average rate of change of temperature increase over time from $t = 5\ s$ to $t = 45\ s$.

b) Find the average rate of change of temperature increase over time from $t = 45\ s$ to $t = 80\ s$.

c) Using the table, find the instantaneous rate of change of temperature increase over time at $t = 45\ s$.

d) Describe how the temperature increased over the 80 seconds.

2) Given the graph of the function $f(x)$ shown on the right, estimate the point(s) / interval(s)/ x values that satisfy each set of criteria for:



a) function is positive

b) function is decreasing

c) instantaneous rate of change is constant

d) instantaneous rate of change is increasing

e) first differences are positive

f) second differences are negative

g) removable discontinuity

h) jump discontinuity

i) where the instantaneous rate of change equals the average rate of change for $1 \le x \le 2.5$

j) the function has a turning point

k) the derivative has a zero

l) the derivative is positive

m) the derivative is increasing

n) the derivative is negative and decreasing

o) the derivative has a turning point

p) function is concave down

q) the derivative is undefined.

3) What is meant by rate of change? Distinguish between average rate of change and instantaneous rate of change.

4) Give an example of a function where the first derivative is zero, but the function **does not** have an extrema.

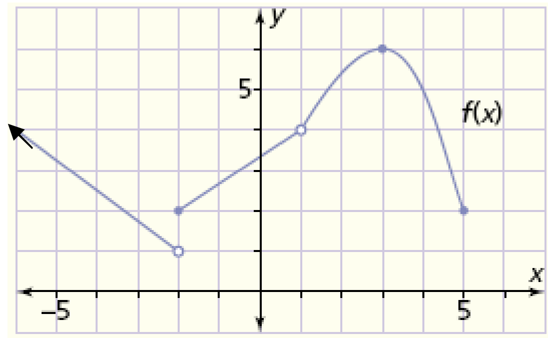
5) Give an example of a function where the first derivative is undefined, but the function **does** have an extrema.

6) The graph of function $f(x)$ is shown on the right, use it to determine each of the following limits.

(a) $\lim\limits_{x \to -2^-} f(x)$ **(b)** $\lim\limits_{x \to -2^+} f(x)$ **(c)** $\lim\limits_{x \to -2} f(x)$

(d) $\lim\limits_{x \to 3} f(x)$ **(e)** $\lim\limits_{x \to 1^-} f(x)$ **(f)** $\lim\limits_{x \to 1^+} f(x)$

(g) $\lim\limits_{x \to 1} f(x)$ **(h)** $\lim\limits_{x \to -\infty} f(x)$



7) Evaluate each limit

a) $\lim\limits_{x \to 2}\left(x^2+5\right)$ b) $\lim\limits_{x \to -1}\left(x^2-1\right)$ c) $\lim\limits_{x \to 2}\dfrac{2x}{x^2-4}$

d) $\lim\limits_{x \to -4}\dfrac{2x^2+5x-12}{x+4}$ e) $\lim\limits_{x \to \infty}\dfrac{1-2x^3-4x^6}{5x^2-2x^6}$ f) $\lim\limits_{x \to \infty}\dfrac{1-2x^3}{5x^2-2x^6}$

g) $\lim\limits_{x \to \infty}\left(3^{-x}\right)$ h) $\lim\limits_{x \to -\infty}\left(e^x-4\right)$

8) Find any point of discontinuity for each function. State the type of discontinuity, and if possible, redefine the function so that it is continuous.

a) $f(x)=\dfrac{x}{(x-5)^2}$ b) $f(x)=\dfrac{x-2}{x^2-8x+12}$ c) $f(x)=\begin{cases} x^3-14, & x \le 0 \\ x^2+5x+14, & x > 0 \end{cases}$

d) $f(x)=\begin{cases} x^3-3x^2+2x+4, & x \le 0 \\ \dfrac{x^2-x+12}{x-3}, & x > 0 \end{cases}$ e) $f(x)=\dfrac{x^2+x-12}{x-3}$

9) Express as a single natural logarithm:

a) $\ln 600+\ln 2-\ln 30$ b) $3\ln 2+\frac{1}{2}\ln 16$ c) $3\ln(2x)+2\ln x-\ln 4x$ d) $2\ln(5\sqrt{x})-\frac{1}{3}\ln(8x)$

10) Evaluate:

a) $\ln e$ b) $-5e^{\ln(\ln e)}$ c) $\ln\left(\dfrac{1}{e^3}\right)$ d) $(-2)^{e^{\ln 3}}$

11) Differentiate

a) $y = 4x^2 - 7x + 8$

b) $f(x) = \sqrt{x} + \sqrt[3]{x}$

c) $p(x) = (-3x^2 - 7x + 8)(4x - 1)$

d) $f(x) = \dfrac{5}{x^2} - \dfrac{4}{x^3}$

e) $f(x) = 4x^{-3}\left(4x - 2x^{-2}\right)$

f) $f(x) = \dfrac{x^2 - 2x + 4}{6}$

g) $q(x) = \dfrac{x+1}{x^2 + 3}$

h) $c(x) = \dfrac{1}{10 - 0.1x^3}$

i) $f(x) = \dfrac{x^3 - x}{2x^2 + x - 1}$

j) $f(x) = \left(4x^2 + 3x\right)^5$

k) $l(x) = \ln\left[\dfrac{(x-1)^{\frac{2}{5}}}{x+1}\right]$

l) $f(x) = \dfrac{(2x+5)^2}{\left(4x^3 - 6x^2\right)^3}$

m) $x^2 + 4y^2 = 6xy$

n) $f(x) = e^{5x}$

o) $g(x) = e^{1-6x}$

p) $h(x) = \ln(2x)$

q) $j(x) = \ln(\ln(x))$

r) $k(x) = \sqrt{\ln x}$

s) $f(x) = \left(6x^2 - 5x\right)^2 (2x-1)^4$

t) $m(x) = 5 \cdot 3^x$

u) $n(x) = 14^{2x+1}$

v) $p(x) = 7^x + x^5$

w) $q(x) = (3)^{4^x}$

x) $r(x) = \sin x$

y) $t(x) = \cos\left(x^2 - 3\right)$

z) $u(x) = \sec x$

12) Determine the first derivative, for each of the following functions, from first principles.

a) $g(x) = 2x + 4$

b) $T(x) = -x^2 + 3x - 7$

c) $W(x) = 4x^3 - 3x - 2$

d) $R(x) = \dfrac{x-1}{x+2}$

e) $S(x) = \sqrt{x - 4}$

13) Determine $\dfrac{d^2y}{dx^2}$

a) $y = -15x^4 + 7x$

b) $y = 5x^{-3} + 10x^3$

c) $y = 15$

d) $x^3 + y^3 = 1$

f) $y = e^x$

g) $y = 9^x$

h) $y = 2000 \cdot \ln x$

i) $y = \cos^2 x$

14) Determine the slope of the tangent line to the curve $y = \cos 2x$ at the point where $x = \dfrac{\pi}{6}$

15) Determine the equation of the tangent line to the curve $n(x) = \dfrac{-4}{x^4 + 3}$ at the point where $x = \frac{1}{3}$.

16) Find the equation of the tangent line to the curve $f(x) = x^2 - 7x + 12$ which is parallel to the line $7x + y - 5 = 0$.

17) Determine the slope of the tangent line to the curve $x^3 + y^3 = y + 21$ at the point where $x = 3$.

18) Find the equation of the tangent line to the curve $f(x) = 2x^4$ that has a slope of 1.

19) Determine the equation of the tangent line to the curve $f(x) = 4x^3 + 12x^2 - 96x$ with the smallest slope.

20) Find the equations of both lines that pass through the point $\left(\frac{-5}{3}, -16\right)$ and are tangent to the parabola $y = x^2 - 8x$.

21) Determine the extrema and the interval(s) over which the function is increasing/decreasing

a) $f(x) = x^4 - 6x^3 - 5x^2$

b) $f(x) = (5x+3)(x^2 - x - 1)$

c) $y = 2x - \ln x$

22) Determine the point(s) of inflection and the intervals of concavity

a) $f(x)=x^3-3x^2-45x$ b) $f(x)=x^4-12x^2$ c) $y=x^2e^{-x^2}$

23) The estimated population of a bacteria colony is $P(t)=20+61t+3t^2$, where the population, *P*, is measured in thousands at *t* hours.

a) What is the estimated population of the colony at 8 hours?
b) What is the average rate of change from 4 hours to 8 hours?
c) What is the instantaneous rate at which the population is changing at 8 hours?

24) The position of an object, moving along a straight line, is given by $s(t)=2t^3-3t^2+5t-6$, where position is in cm and *t* is in seconds.

a) Describe the motion of the object at t = 4 seconds
b) What is the initial position of the object?
c) When is the object at the origin?
d) During what time interval(s) is the object moving to the right?
e) Determine when the object reverses direction.
f) When is the object moving at 3 cm/s to the left?
g) What is the average velocity of the object when $2 \le t \le 5$
h) When is the object speeding up?
i) Determine the position, velocity and acceleration at t = 2 seconds and determine if the object is moving toward or away from the origin.
j) Determine the distance travelled by the object in the first 5 seconds.

25) The volume of a cube with side length *x* is $V(x)=x^3$. Find the rate of change of volume with respect to side length, if the surface area of the cube is 54 cm^2.

26) The concentration, C, of a drug injected into the bloodstream t hours after injection can be modelled by $C(t)=\frac{t}{4}+2t^{-2}$. Determine when the concentration of the drug is increasing and when it is decreasing.

27) A park ranger has 600 m of floating rope. She is going to enclose a rectangular swimming area, using the beach as one border of the area. Find the maximum area that can be enclosed and the corresponding dimensions.

28) The landlord of a 50-unit apartment building is planning to increase the rent. Currently, residents pay $850/month and all units are occupied. A real estate agency advises that every $100 increase in rent will result in 10 vacant units. Should the landlord change the rent in order to maximize the revenue? If so, what would be the new rent?

29) A cylindrical, open container is to be made to hold 2L (2000 cm^3) of liquid. The circular bottom is to be stamped out from a rectangular sheet. The unused metal form this rectangle is discarded. The material for the bottom costs $0.005/cm^2. The material for the sides costs $0.10/cm^2. What are the dimensions of the least expensive container? If the material could be bought in circular “sheets”, what would be the dimensions of the least expensive container?

30) Ron is planting a garden all around a rectangular patio. The garden will be 5 m wide, except at the diagonals. The area of the patio must be 150 m^2. Find the overall dimensions of the garden and patio in which the area of the garden is a minimum. (hint: the patio around the garden is like the margins around the printed portion of a page.)

31) Katrina is in a boat 2 km from a straight shoreline and wants to reach a point on the shore 10 km north of her present position. She can row at 5 km/h and jog at 8 km/h. Calculate the position on the shoreline to which she should row to reach the destination in the shortest time.

32) William owns an oil well located 400 m from a road. William wants to connect the well to a storage tank 1200 m down the road from the well. It costs $35/m to lay pipe along the road and $50/m to lay it elsewhere. How should the pipeline be laid to minimize the total cost?

33) The atmospheric pressure, y, varies with the altitude, *x* kilometres, above the Earth. For altitudes up to 10 km, the pressure in millimtres of mercury (mm Hg) is given by $y = 760e^{-0.125x}$.

a) What is the atmospheric pressure 5 km above the Earth?
b) Determine the rate at which the atmospheric pressure is changing when you are 9 km above the Earth.

34) Find the radius and height of a right-circular cylinder of largest volume that can be inscribed in a right-circular cone with radius 12cm and height 20 cm.

35) A pipe is to be carried around a corner from a hallway that is 4 m wide into a hallway that is 4 m wide. What is the maximum length of the pipe?

36) A plank is used to reach over a 2 m fence and support a wall that is 1 m behind the fence. What is the length of the shortest plank that can be used? How far from the fence should this plank be placed?

37) Sketch a function with the following properties:

a) $\lim_{x\to\pm\infty} f(x) = 3$
b) $\lim_{x\to -2} f(x) = \pm\infty$ [no limit] and $\lim_{x\to 2} f(x) = \pm\infty$ [no limit]
c) $f'(x) < 0$ for $x < 0$ only
d) $f''(x) > 0$ for $-2 < x < 2$ only
e) $f(-4) = -3$ and $f(4) = -2$

38) Sketch the graph of a polynomial function that satisfies the following contitions:

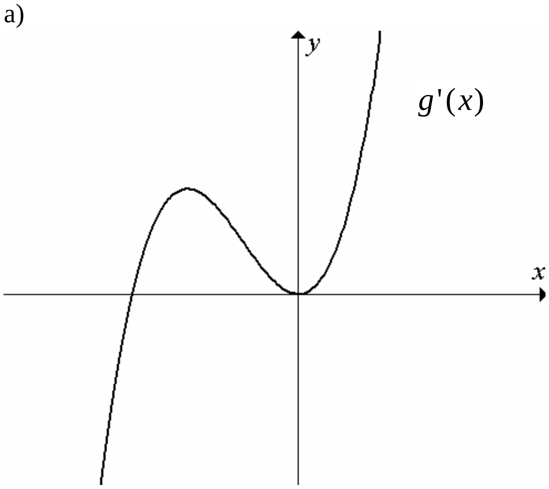
☞ $f(x) \le 0$ when $x \le -3$ and when $x \ge 0$
☞ first differences are positive when $x < -2$
☞ second differences are negative when $x < -1$ and when $x > 0$

39) Create a graph with all of the following characteristics. Determine an equation of a function that models your graph.

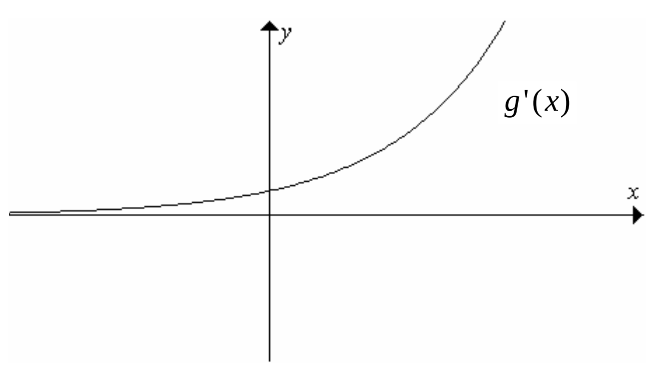
$f'(x) > 0$ for $x < -2$ and $-2 < x < 0$
$f''(x) > 0$ for $x < -2$ and $x > 3$
$\lim_{x\to\pm\infty} f(x) = 2$
$\lim_{x\to 3^+} f(x) = \infty$
$\lim_{x\to -2^+} f(x) = -\infty$
$f'(x) < 0$ for $0 < x < 3$ and $x > 3$
$f''(x) < 0$ for $-2 < x < 3$
$\lim_{x\to 3^-} f(x) = -\infty$
$\lim_{x\to -2^-} f(x) = \infty$
$f(0) = 0$

40) Given the graph of the first derivative of the function $g(x)$, sketch the graph of the function and of the second derivative.

a)



b)



41) Analyse each of the following functions and use your analysis to sketch the graph of the function.

a) $f(x)=\dfrac{5x}{(x-1)^2}$, $f'(x)=\dfrac{-5(x+1)}{(x-1)^3}$, $f''(x)=\dfrac{10(x+2)}{(x-1)^4}$

b) $g(x)=(x-1)^3-1$, $g'(x)=3(x-1)^2$, $g''(x)=6(x-1)$

c) $k(x)=\sqrt{x+5}$, $k'(x)=\dfrac{1}{2\sqrt{x+5}}$, $k''(x)=\dfrac{-1}{4(x+5)^{\frac{3}{2}}}$

d) $j(x)=(x-1)^{\frac{1}{3}}+2$, $j'(x)=\dfrac{1}{3}(x-1)^{\frac{-2}{3}}$, $j''(x)=\dfrac{-2}{9}(x-1)^{\frac{-5}{3}}$

e) $h(x)=x\sqrt{9-x^2}$, $h'(x)=\dfrac{9-2x^2}{\sqrt{9-x^2}}$, $h''(x)=\dfrac{2x^3-27x}{(9-x^2)^{\frac{3}{2}}}$

f) $m(x)=\dfrac{(x-1)^2}{(x+1)^3}$, $m'(x)=\dfrac{-x^2+6x-5}{(x+1)^4}$, $m'(x)=\dfrac{2(x^2-10x+13)}{(x+1)^5}$

g) $n(x)=xe^{-3x}$, $n'(x)=(-3x+1)e^{-3x}$, $n''(x)=3(3x-2)e^{-3x}$

h) $l(x)=\ln(x^2-3)$, $l'(x)=\dfrac{2x}{x^2-3}$, $l''(x)=\dfrac{-6}{(x^2-3)^2}$