OPTIMIZATION

Strategy for solving Optimization problems:

1. **Recognize the Problem Type**

Look for words like <u>greatest</u>, <u>most</u>, <u>fewest</u>, <u>least</u>, <u>maximum</u>, <u>minimum</u>, <u>largest</u>, <u>smallest</u>, etc. The word <u>best</u> (or <u>optimal</u>) could also be used-it could mean biggest(if talking about profit, for example) or smallest (if talking about costs, for example).

<u>*Identify the variable*</u> that is to be maximized or minimized.

2. **Establish a Formula**

This may be a formula that you are expected to know (such as the area of a circle, Pythagorean theorem, etc.), or it may be provided as part of the problem, or you may be able to develop it using information given in the problem. It is often helpful to draw a diagram.

Circle: $A = \pi r^2$, $C = 2\pi r$ — Rectangle: $P = 2l + 2w$, $A = lw$

Sphere: $V = \frac{4}{3}\pi r^3$, $SA = 4\pi r^2$ — Cone: $V = \frac{1}{3}\pi r^2 h$, $SA = \pi r^2 + \pi r\sqrt{h^2 + r^2}$

Cylinder: $V = \pi r^2 h$, $SA = 2\pi r^2 + 2\pi rh$

Rectangular Solid: $V = lwh$, $SA = 2hw + 2lw + 2hl$

3. **Prepare for Differentiation**

Before you differentiate it is <u>imperative</u> that you eliminate any extra variables from the formula. There can only be two variables remaining, the one that you are attempting to optimize and one other. It is also a good idea to put the formula in a form that will make the process of differentiation as easy as possible.

To eliminate extra variables:

Substitute using relationships that are provided in the problem (such as $h = 3r$), or relationships that are common knowledge (such as $d = 2r$).

<u>If any variable remains constant</u> in this problem, you should substitute the value at this point. <u>**You must not substitute values which are not constant**</u>.

4. **Take the Derivative**

Differentiate the variable to be optimized with respect to the second variable kept in the problem. Use appropriate rules (product, quotient, chain, implicit, etc.) It is probably best to use $\frac{dV}{dr}$ form.

5. **Find Maximum/Minimum Points**

Let the derivative equal zero and find sign change points in the usual way. Test these values to determine which is a maximum and which is a minimum-use the first derivative sign line to do this. (For most problems, it will be obvious which is which, but you should do the test for sake of form and completeness).

6. **Answer the Question**

Check the problem to see what information is requested. It could be the actual maximum or minimum value, or the value for the second variable at which the maximum or minimum occurs. It could be both!!! Make a formal statement of this answer.

A sheet of cardboard will be used to make an open-topped box. If the cardboard has dimensions 18 in by 24 in, determine the dimensions of the box that will have greatest volume.

It is noon and you are in your canoe 3 km off the shore of a river and you want to get to a location 12 km down the straight shoreline from the nearest point on the shore. If you can canoe at 4 km/h and run at 5 km/h, where on the shoreline should you canoe to in order to make it to the location in the least amount of time. What is the least amount time it takes to get to the location?

Practice:

1) The number of cells in a tissue culture is given by: $n(t)=1000+360t-49t^2+6t^3$, where n is the number of cells and t is the time in hours.
   a) How many cells are present after 1 hour? {1317 cells}
   b) At what rate is the number of cells changing after 3 hours? {228 cells/hour}
   c) Determine at what time the number of cells will be increasing at the rate of 320 cells/hour? {27 minutes and 5 hours}

2) A carpenter is building an open box with a square base for holding firewood. The box must have a surface area of 8 m^2. What dimensions will yield the maximum volume? What is the maximum volume? {side length 1.63m, height, 0.82 m volume 2.18 m^3}

3) A local bus company carries 600 passengers a day at a flat rate of $1.50. If the fare is raised by $0.05, 40 fewer people travel by bus. What fare should be charged to produce the greatest gross income for the bus company? {$1.10-$1.15}

4) A builder must subdivide four adjacent rectangular building lots, each of 540 m^2, and enclose them with fences. If the back and side fencing costs $10/ meter and the front fencing costs $20/ meter, find the least cost he must incur? {$3600}

5) Find the length of the sides of the isosceles triangle of greatest area that has a perimeter of 18 cm. {6cm/side}

6) A student in an advanced typing class improves typing speed as the weeks pass. It has been shown that the typing speed (number of words/minute) W is given by $W(t)=\dfrac{60t+90}{t+6}$ where t is the time in weeks since the beginning of the course.
   a) How many words/minute would you expect a student to be able type after one week? {21-22 wpm}
   b) What is the average rate of increase in the typing speed during the first three weeks? {5wpm/week}
   c) How fast is the number of words/minute increasing after 4 weeks? {2.7 wpm/week}
   d) When would the typing speed be increasing at the rate of $1\frac{7}{8}$ words/minute? {6 weeks}

7) A printed page of total area 320 m^2 has top and side margins of 2 cm and a bottom margin of 3 cm. Find the dimensions of the page that make the area of print a maximum. {16 cm by 20 cm}

8) A Norman window contains a rectangular pane of glass surmounted with a semi-circular pane with diameter equal to the topmost side of the rectangle. Find the dimensions of a Norman window of perimeter 12 m which has the greatest possible area of glass. {r = 1.68 m}

9) A canoeist is 300m offshore and wishes to land and then walk to a distant point on the straight shoreline. If she can paddle at 3 km/h and walk at 5 km/h, where should she land to minimize her travel time? {225 m along shore}

10) Find the length of the longest pole which can be carried around the corner from one hallway which is 2.5 m wide to a perpendicular hallway 3 m wide. Assume the pole must be parallel to the ground. {7.78 m}

11) A piece of wire 100 cm long is divided into two pieces. One piece is used to form a circle and the other a square. Find the lengths of wire cut so that the combined area of the circle and the square is a minimum. {44 cm and 56 cm}

12) Find the circular cylinder of greatest volume that can be inscribed in a right circular cone with altitude 20 cm and diameter of base 10 cm. $\{h = 6\tfrac{2}{3} cm, r = 3\tfrac{1}{3} cm\}$

13) A ball of paper falls from the top of the CN tower. Its height, *h*, above the ground in metres after *t* seconds is given by $h(t) = 603 - \sqrt{t^2 + 9}$, $t \geq 0$.

a) Determine the height of the ball of paper after four seconds.
b) Find the average rate of change of the ball of paper from $t = 10$ seconds to $t = 20$ seconds.
c) Find the instantaneous rate of change at 4 seconds.

14) A sailing ship is 25 Nautical miles due north of a drifting derelict vessel. If the ship sails south at 4 knots while the derelict drifts east at 3 knots, find the distance of closest approach of the two ships. {15 Nautical miles}

15) A company can sell 5000 chocolate bars a month at $0.50 each. If they raise the price to $0.70, sales drop to 4000 bars per month. The company has fixed costs of $1000 per month and $0.25 for manufacturing each bar. What price will maximize the profit? {$0.90}

16) A wall is 1.8 m high and 1.2 m from a building. Find the length of the shortest ladder that will touch the building, the top of the wall, and the ground beyond the wall. {4.2}

17) A company determines that the cost, in dollars, of producing x item is given by $C(x) = 280000 + 12.5x + 0.07x^2$

a) Find the average cost of producing 1000 items. (Average Cost = $\frac{C(x)}{x}$) {$362.50/item}
b) Find the marginal cost of producing 1000 items. (Marginal Cost= $\frac{d}{dx}C(x)$) {$152.50/item}
c) At what production level will the average cost be minimized? {2000 items}
d) What is the minimum average cost? {$292.50}

18) Find the circular cylinder of greatest volume that can be inscribed in a sphere of radius 6 cm. {h = 6.9 cm, r = 4.9 cm}