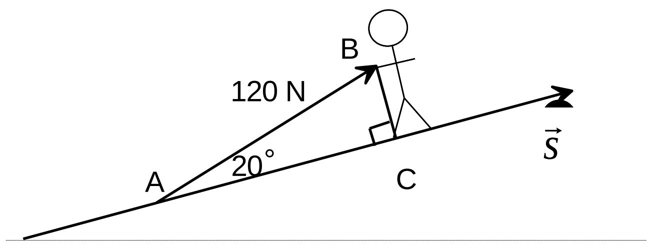
<u>The Dot Product and Applications</u>

There are situations in physics where vectors can be combined in ways other than addition. The scalar quantity **work** is defined as the displacement travelled multiplied by the magnitude of the applied force in the direction of motion. The units are Newton-metres also known as joules (J).

Ex. Aaron is pulling his sled up a hill with a force of 120 N at an angle of 20° to the surface of the hill. The hill is 100 m long. Find the work that Aaron performs.



We need the magnitude of the component of the force along the surface of the hill: AC.

$$\cos 20^\circ = \frac{AC}{AB}$$
$$AC = AB\cos 20^\circ$$
$$= 120\cos 20^\circ$$

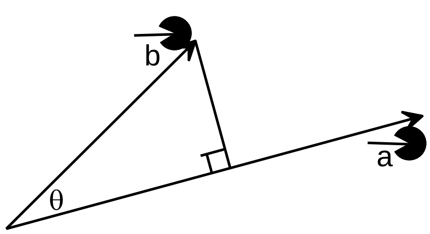
From the definition of work,

$$work = |\vec{s}|(AC)$$
$$= (100)(120\cos 20^\circ)$$
$$\doteq 11276J$$

$\vec{s}$ is the displacement, $\overrightarrow{AC}$ is the component of $\overrightarrow{AB}$ that is the direction of $\vec{s}$

The work done by Aaron in pulling his sled up the hill is 11276 J.

This new product is called the **dot product**.



For non-zero vectors $\vec{a}$ and $\vec{b}$, we define the **dot product** as:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$|\vec{a}|$ is the displacement,
$|\vec{b}|\cos\theta$ is the component of $\vec{b}$ that is the direction of $\vec{a}$.

where θ is the angle between $\vec{a}$ and $\vec{b}$.

This is a scalar quantity and is sometimes called the **<u>scalar product</u>**.

Applications of the Dot Product

Definition: The dot product of two vectors $\vec{u}$ and $\vec{v}$ is $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

This is a scalar quantity and is sometimes called the scalar product.
Of course this works in three space as well.

Properties of the Dot Product

$$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$$
$$a(\vec{u}\cdot\vec{v}) = (a\vec{u})\cdot\vec{v} = \vec{u}\cdot(a\vec{v}),\ a\in\Re$$
$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$$
$$\vec{u}\cdot\vec{u} = |\vec{u}|^2$$

Aside:

$$\hat{i}\cdot\hat{i} = 1$$
$$\hat{j}\cdot\hat{j} = 1$$
$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = 0$$

Dot Product in Algebraic Form in R^2

Given $\vec{u} = (u_1, u_2)$ $\vec{v} = (v_1, v_2)$

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= (u_1,u_2)\ .\ (v_1,v_2) \\
&= (u_1\hat{i}+u_2\hat{j})\ \cdot\ (v_1\hat{i}+v_2\hat{j}) \\
&= u_1\hat{i}\ \cdot\ (v_1\hat{i}+v_2\hat{j}) + u_2\hat{j}\ \cdot\ (v_1\hat{i}+v_2\hat{j}) \\
&= u_1v_1\hat{i}\cdot\hat{i} + u_1v_2\hat{i}\cdot\hat{j} + u_2v_1\hat{j}\cdot\hat{i} + u_2v_2\hat{j}\cdot\hat{j} \\
&= u_1v_1 + u_2v_2
\end{aligned}$$

Dot Product in Algebraic Form in R^3

Given $\vec{u} = (u_1, u_2, u_3)$ $\vec{v} = (v_1, v_2, v_3)$

$$\begin{aligned}
\vec{u}\cdot\vec{v} &= (u_1,u_2,u_3).(v_1,v_2,v_3) \\
&= (u_1\hat{i}+u_2\hat{j}+u_3\hat{k})\cdot(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) \\
&= u_1\hat{i}\cdot(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) + u_2\hat{j}\cdot(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) + u_3\hat{k}\cdot(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) \\
&= u_1v_1\hat{i}\cdot\hat{i} + u_1v_2\hat{i}\cdot\hat{j} + u_1v_3\hat{i}\cdot\hat{k} + u_2v_1\hat{j}\cdot\hat{i} + u_2v_2\hat{j}\cdot\hat{j} + u_2v_3\hat{j}\cdot\hat{k} + u_3v_1\hat{k}\cdot\hat{i} + u_3v_2\hat{k}\cdot\hat{j} + u_3v_3\hat{k}\cdot\hat{k} \\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned}$$

Theorem: Two non-zero vectors are perpendicular if and only if $\vec{u}\cdot\vec{v} = 0$.

Applications of the Dot Product

#1 Finding the angle between two vectors in component form.

Find the angle between $\vec{u} = (-1,5,2)$ $\vec{v} = (2,-3,1)$

#2 Finding a vector perpendicular to a given vector.

Ex. Find a vector perpendicular to $\vec{w} = (-4,\ 7)$

Ex. Find a vector perpendicular to $\vec{w} = (-4,7,3)$

#3 Finding the dot product of the sum or difference of the same two vectors.

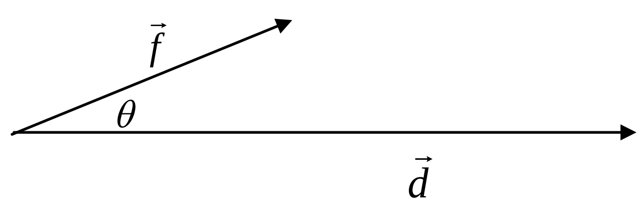
Ex. Given $\vec{u} = (-1,\ 5)$ $\vec{v} = (2,\ -3)$ Find $(3\vec{u} - \vec{v}) \cdot (2\vec{u} + \vec{v})$

#4 Finding the work done.

Work = Displacement • Force along the displacement

$Work = |\vec{f}||\vec{d}|\cos\theta$

scalar measured in Joules

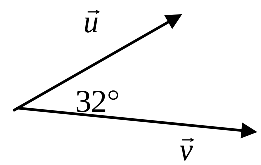


Homework on dot product and work:

1. Given vectors $\vec{u}$ *and* $\vec{v}$ as shown, find: $\vec{u}\cdot\vec{v}$

a) $\vec{v}\cdot\vec{u}$ {144.2}

b) $(2\vec{u}-\vec{v})\cdot(3\vec{u}+3\vec{v})$ {165.5}



$|\vec{u}|=10, |\vec{v}|=17$

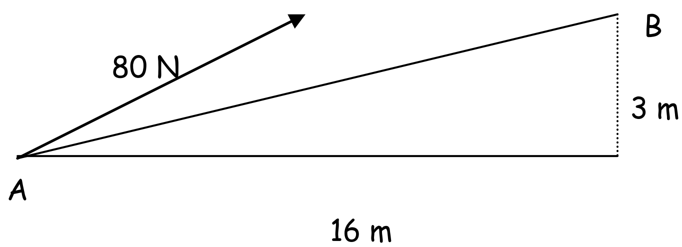
2. Given $\vec{y}=2\hat{i}+2\hat{j}, \quad \vec{x}=(-1,4), \quad \vec{z}=(3,-2)$
$A(-1,-3), \quad B(-3,-2)$

Find: a) $\vec{y}\cdot\vec{x}$ {6}

b) the angle between $\vec{x}$ *and* $\vec{z}$ {137.7 degrees}

c) $(2\vec{x}-\vec{y})\cdot(5\vec{x}+\vec{y})$ {144}

3. Calculate the work done* if a box is pulled from point A to point B (as illustrated) by a force of 80 N applied at an angle of 30 degrees to the horizontal. (*find work done on box) {1230 Joules}



4. Given $\vec{y}=2\hat{i}+2\hat{j}+7\hat{k}, \quad \vec{x}=(-1,4,6), \quad \vec{z}=(3,-2,-3), \quad A(-1,-3,5), \quad B(-3,-2,4)$

Find: a) $\vec{y}\cdot\vec{x}$ {48}

b) the angle between $\vec{x}$ *and* $\vec{z}$ {149 degrees }

c) $(2\vec{x}-\vec{y})\cdot(5\vec{x}+\vec{y})$ {329}

5. Given W(-1,5,3), X(6,-2,-2) and Y(-2,-7,9), find:

a) the co-ordinates of the point Z if WXYZ is a parallelogram. {Z(-9,0,14)}

b) the angle in the parallelogram at X. {61.8 degrees }