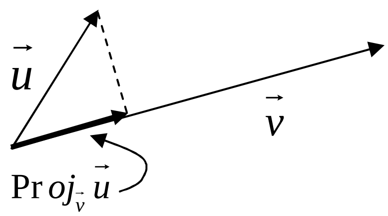
Determining Distances for Lines in Two Space and Three Space

Recall:

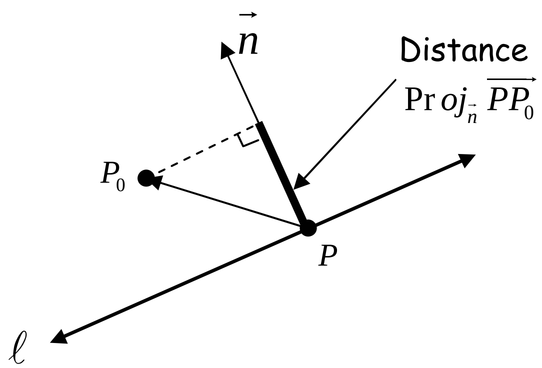


$$\left|\mathrm{Proj}_{\vec{v}}\vec{u}\right| = \left|\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|}\right|$$ is the length of the projection

TWO SPACE

a) **Finding the distance between a point and a line in two space:**

- Find the magnitude of the projection of the vector between the point and a point on the line onto the normal for the line. This will be the distance between the point and the line.



P_0 – given point
P – point on line (you pick)

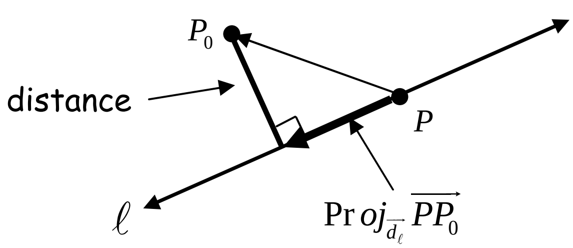
In two space, a line has a normal and the distance is equal to the magnitude of the projection of $\overrightarrow{PP_0}$ onto $\vec{n}$

$$dist = \left|\mathrm{Proj}_{\vec{n}}\,\overrightarrow{PP_0}\right| = \left|\frac{\vec{n}\bullet\overrightarrow{PP_0}}{|\vec{n}|}\right|$$

This works in R^2 only!

OR

- Find the magnitude of the projection of the vector between the point and a point on the line onto the direction vector for the line. Use this and Pythagorean's Theorem to get the distance between the point and the line.



Pythagorean Theorem:

$$|dist|^2 + \left|\mathrm{Proj}_{\vec{d}_\ell}\,\overrightarrow{PP_0}\right|^2 = \left|\overrightarrow{PP_0}\right|^2$$

$$|dist|^2 = \left|\overrightarrow{PP_0}\right|^2 - \left|\mathrm{Proj}_{\vec{d}_\ell}\,\overrightarrow{PP_0}\right|^2$$

b) **Finding the distance between parallel lines in two space:**

- Select a point on one line and find the distance between a point and a line as outlined above.

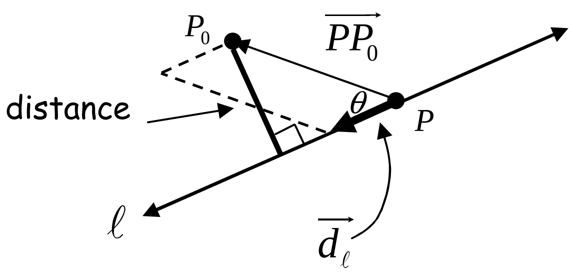
THREE SPACE

a) **Finding the distance between a point and a line in three space:**

- Find the magnitude of the projection of the vector between the point and a point on the line onto the direction vector for the line. Use this and Pythagorean's Theorem to get the distance between the point and the line. Same as in 2-space as it relies on a direction vector and not a normal.

OR

- Find the magnitude of the cross product of the vector between the point and a point on the line with the direction vector for the line. Divide this number by the magnitude of the direction vector. This will give you the distance between the point and the line. This is the height of the parallelogram.



We have

$$\left|\overrightarrow{PP_0}\times\vec{d}_\ell\right| = base\times height$$

$$= \left|\vec{d}_\ell\right|\times dist$$

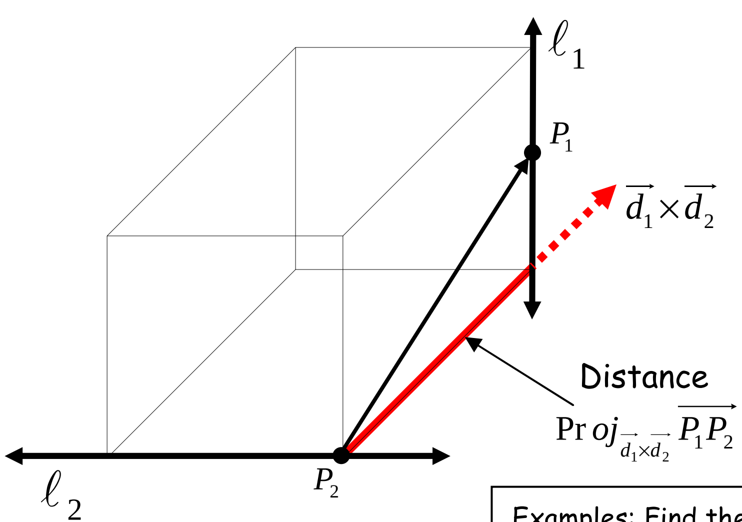
$$\therefore dist = \frac{\left|\overrightarrow{PP_0}\times\vec{d}_\ell\right|}{\left|\vec{d}_\ell\right|}$$

b) **Finding the distance between parallel lines in three space:**

- Select a point on one line and find the distance between a point and a line as outlined above.

c) **Finding the distance between skew lines in three space:**

- Find the magnitude of the projection of the vector between the point on one line and a point on the other line onto the vector which is perpendicular to both lines.



The shortest distance occurs on a line that is $\perp$ to both ℓ_1 and ℓ_2.

$\therefore$ The direction vector for this line is given by $\vec{d}_1\times\vec{d}_2$

$$\therefore dist = \left|\mathrm{Proj}_{\vec{d}_1\times\vec{d}_2}\overrightarrow{P_1P_2}\right|$$

$$= \left|\frac{\left(\vec{d}_1\times\vec{d}_2\right)\bullet\overrightarrow{P_1P_2}}{\left|\vec{d}_1\times\vec{d}_2\right|}\right|$$

Examples: Find the distance from:

1) Point (-3,7) to the line $2x + y + 5 = 0$ {2.68}

2) Point (1,-1,2) to the line $\ell : (x, y, z) = (3,5,-3) + k(1,1,-1)$ {2.94}

3) $\ell_1 : (x, y, z) = (1,-3,2) + k(1,-1,2)$
$\ell_2 : (x, y, z) = (3,1,-4) + t(3,1,-4)$ {1.83}