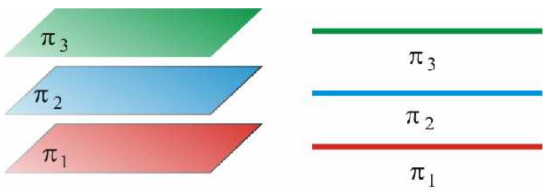
Intersection of Three Planes

Lets consider three planes given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$
$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$
$$\pi_3 : A_3x + B_3y + C_3z + D_3 = 0$$

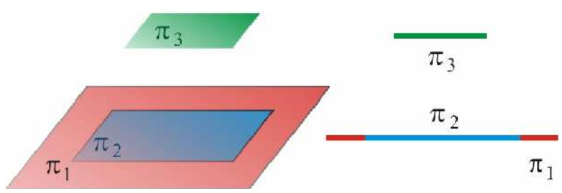
> The intersection of these planes is related to the solution of this 3 x 3 system of equations. Use substitution/elimination to create a 2 x 2 system then solve this system.

Case 1 – Three parallel planes



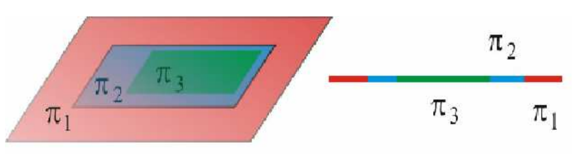
- No intersection
- Inconsistent
- No solution to system of equations
- A, B, C are proportional but A, B, C, D are not proportional
- Normals are scalar multiples
  $\vec{n}_{\pi_1} = m\vec{n}_{\pi_2} = k\vec{n}_{\pi_3}$
- solving 3 x 3 system results in false statement like $0 = 1$

or



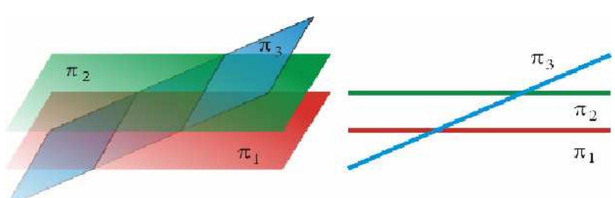
- Also possible, two coincident and third parallel. A, B, C, D proportional for two but not for third.

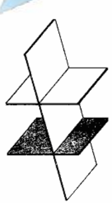
Case 2: Three identical planes



- The intersection is a plane
- Consistent and dependent on two parameters
- A, B, C, D are proportional for all equations
- Normals are scalar multiples
  $\vec{n}_{\pi_1} = m\vec{n}_{\pi_2} = k\vec{n}_{\pi_3}$
- Infinite number of solutions to system $0 = 0$

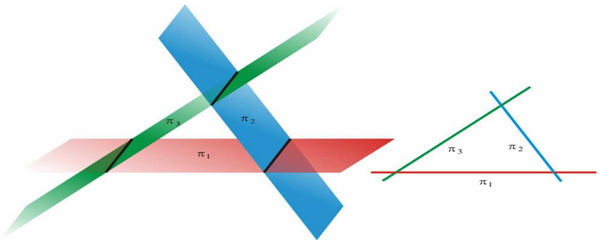
Case 3: Two Parallel lines with one cutting through (H configuration)

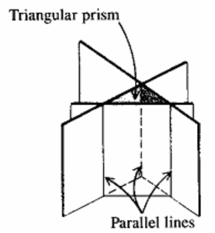




- No intersection
- Inconsistent
- No solution to system of equations
- A, B, C are proportional for two planes
- Only two of the normals are scalar multiples
  $\vec{n}_{\pi_1} = m\vec{n}_{\pi_2} \neq k\vec{n}_{\pi_3}$
- solving 3 x 3 system results in false statement like $0 = 1$

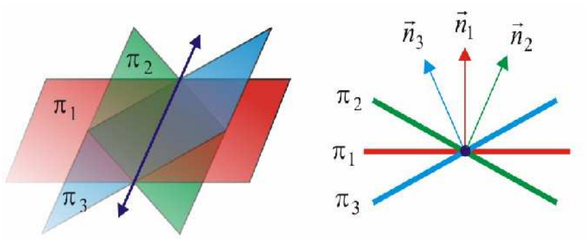
Case 4: Triangular Prism





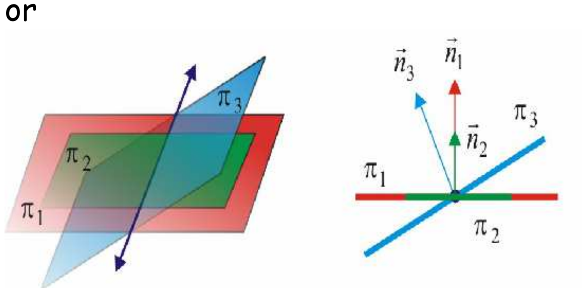
- No intersection of all three planes
- Inconsistent
- No solution to system of equations
- A, B, C are not proportional
- No normals are collinear
- Normals are coplanar $\left(\vec{n}_{\pi_1} \times \vec{n}_{\pi_2}\right) \bullet \vec{n}_{\pi_3} = 0$
- solving 2 x 2 system results in false statement like $0 = 1$. This implies any two planes intersect but not all three.

Case 5: Spine of a book



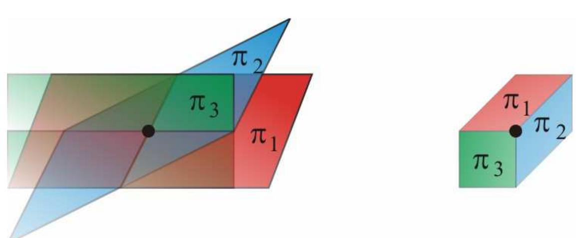
- Line of intersection
- Consistent and dependent on one parameter
- No solution to system of equations
- A, B, C are not proportional
- No normals are collinear
- Normals are coplanar $\left(\vec{n}_{\pi_1} \times \vec{n}_{\pi_2}\right) \bullet \vec{n}_{\pi_3} = 0$
- solving 2 x 2 system results $0 = 0$.

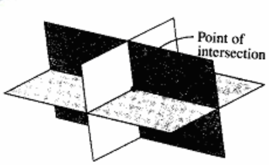
or



- Two scalar equations are equivalent. A, B, C, D are proportional for these two equations.
- Use two non-equivalent equations to find intersection

Case 6: Corner of the room





- Point of intersection
- Consistent and independent
- There is a solution to system of equations
- A, B, C are not proportional
- No normals are collinear $\vec{n}_{\pi_1} \neq m\vec{n}_{\pi_2} \neq k\vec{n}_{\pi_3}$
- No normals are coplanar $\left(\vec{n}_{\pi_1} \times \vec{n}_{\pi_2}\right) \bullet \vec{n}_{\pi_3} \neq 0$
- You will get a unique value for x, y, z when you solve the 3 x 3 system.

Examples:

For each set of three planes determine the intersection (if any). For each set classify the system as consistent or inconsistent, and (if possible) dependent or independent. For each set state the geometrical interpretation between the planes (what case is it?).

$$SET\ 1: \begin{cases} \pi_1 : x - 3y - 2z + 9 = 0 \\ \pi_2 : 2x - 5y + z - 3 = 0 \\ \pi_3 : -3x + 6y + 2z - 8 = 0 \end{cases}$$

$$SET\ \ 2: \begin{cases} \pi_1 : x + y + 2z + 2 = 0 \\ \pi_2 : 3x - y + 14z - 6 = 0 \\ \pi_3 : x + 2y + 5 = 0 \end{cases}$$

$$SET\ \ 3: \begin{cases} \pi_1 : x - y - 2z - 1 = 0 \\ \pi_2 : 2x - 2y - 4z - 2 = 0 \\ \pi_3 : -4x + 4y + 8z + 4 = 0 \end{cases}$$

$$SET\ \ 4: \begin{cases} \pi_1 : x + 2y + 3z - 1 = 0 \\ \pi_2 : 2x + 4y + 6z + 1 = 0 \\ \pi_3 : -x - 2y - 3z - 3 = 0 \end{cases}$$

$$SET\ \ 5: \begin{cases} \pi_1 : x + y - z - 1 = 0 \\ \pi_2 : x + y + z + 2 = 0 \\ \pi_3 : -2x - 2y + 2z - 3 = 0 \end{cases}$$

$$SET\ \ 6: \begin{cases} \pi_1 : 2x + y + z - 1 = 0 \\ \pi_2 : -x + y + z + 1 = 0 \\ \pi_3 : x + y + z = 0 \end{cases}$$

$$SET\ \ 7: \begin{cases} 2x - y + z = 9 \\ 3x + y - z = -4 \\ x + y - 2z = -11 \end{cases}$$

$$SET\ \ 8: \begin{cases} 2x - y + z = 5 \\ 4x - 2y + 2z = 6 \\ -6x + 3y - 3z = -15 \end{cases}$$

Intersection of three planes - Homework

1. For each set of three planes determine the intersection (if any). For each set classify the system as consistent or inconsistent, and (if possible) dependent or independent. For each set state the geometrical interpretation between the planes (what case is it?).

$$SET\ 1: \begin{cases} \pi_1 : x + 2y + 3z - 4 = 0 \\ \pi_2 : 2x - y + 4z + 7 = 0 \\ \pi_3 : 3x - 14y + z + 48 = 0 \end{cases}$$

$$SET\ 2: \begin{cases} \pi_1 : 2x - 3y + z = 22 \\ \pi_2 : x + y - z = -9 \\ \pi_3 : x - y + 3z = 21 \end{cases}$$

$$SET\ 3: \begin{cases} \pi_1 : 2x - y - z - 4 = 0 \\ \pi_2 : 3x - 2y + 3z - 5 = 0 \\ \pi_3 : 13x - 8y + 7z + 7 = 0 \end{cases}$$

$$SET\ 4: \begin{cases} \pi_1 : 5x - y + z - 19 = 0 \\ \pi_2 : 2x + y - 4z - 3 = 0 \\ \pi_3 : 3x - 2y + 2z - 10 = 0 \end{cases}$$

$$SET\ 5: \begin{cases} \pi_1 : 2x - y - z - 4 = 0 \\ \pi_2 : 3x - 2y + 3z - 5 = 0 \\ \pi_3 : 4x - 2y - 2z + 7 = 0 \end{cases}$$

$$SET\ 6: \begin{cases} \pi_1 : 5x - y + z - 9 = 0 \\ \pi_2 : 2x - y + 4z + 15 = 0 \\ \pi_3 : 3x - 2y + 2z + 3 = 0 \end{cases}$$

$$SET\ 7: \begin{cases} \pi_1 : x - y + z - 9 = 0 \\ \pi_2 : 2x - 2y + 2z - 18 = 0 \\ \pi_3 : 3x - 3y + 3z - 27 = 0 \end{cases}$$

$$SET\ 8: \begin{cases} \pi_1 : x - y + z - 9 = 0 \\ \pi_2 : 2x - 2y + 2z + 18 = 0 \\ \pi_3 : 3x - 3y + 3z - 2 = 0 \end{cases}$$

2. In the following system of equations, k is a real number.

$\pi_1 : x - 2y + z = 4$

$\pi_2 : x - y - z = 3$

$\pi_3 : x + y + kz = 1$

For what value(s) of k does the system

i) have exactly one solution?

ii) have an infinite number of solutions?