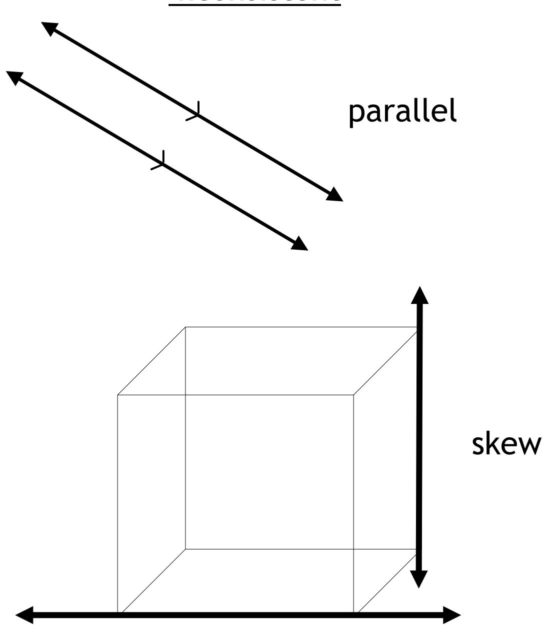
Intersection of Lines in Two and Three Space

Finding the point of intersection of two lines is also referred to as solving a system of linear equations.

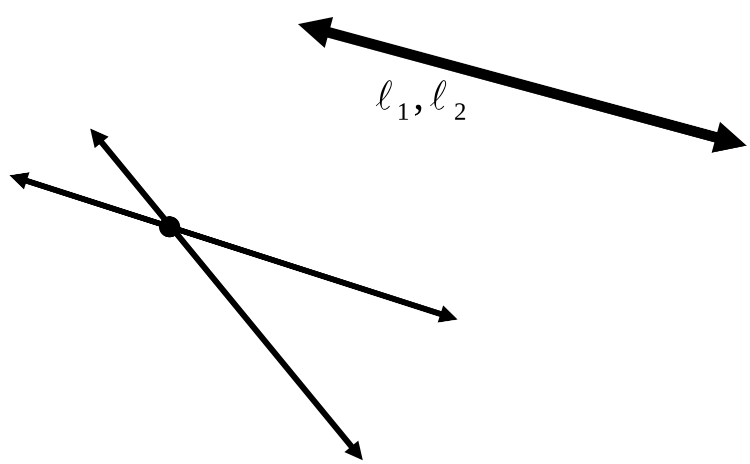
The solution can be classified in the following manner:

- System of equations
  - Inconsistent (no solution)
    - in R^2: lines are || but not the same
    - in R^3: lines are || but not same or lines are skew
  - Consistent (solution(s))
    - Independent (one solution)
      - in R^2: lines are not ||
      - in R^3: lines are not || or skew
    - Dependent (∞ 'ly many solutions)
      - in R^2 and in R^3, lines are || and equal

Inconsistent



Consistent



Explore the intersection of the following systems

a) $\ell_1 : (x, y, z) = (-2,0,-3) + t(5,1,3)$
$\ell_2 : (x, y, z) = (5,8,-6) + s(-1,2,-3)$

b) $\ell_1 : x + 2y = 3$
$\ell_2 : (x, y) = (5,-1) + k(6,-3)$

c) $\ell_1 : \vec{r} = (2,-1,0) + t(1,2,-3)$
$\ell_2 : \dfrac{x+1}{-2} = \dfrac{y-1}{1} = \dfrac{z-2}{1}$

d) $\ell_1 : (x, y, z) = (-2,0,-3) + t(2,1,3)$
$\ell_2 : (x, y, z) = (2,2,3) + s(-4,-2,-6)$

Investigation:

a) $\ell_1 : (x, y, z) = (-2,0,-3) + t(5,1,3)$
$\ell_2 : (x, y, z) = (5,8,-6) + s(-1,2,-3)$ → In parametric form: $\ell_1 : \begin{aligned} x &= -2+5t \\ y &= t \\ z &= -3+3t \end{aligned}$, $\ell_2 : \begin{aligned} x &= 5-s \\ y &= 8+2s \\ z &= -6-3s \end{aligned}$

- First, set coordinates equal to each other

$$\begin{aligned} -2+5t &= 5-s \\ t &= 8+2s \\ -3+3t &= -6-3s \end{aligned} \quad \rightarrow \quad \begin{aligned} 5t+s &= 7 \quad ① \\ t-2s &= 8 \quad ② \\ 3t+3s &= -3 \quad ③ \end{aligned}$$

- Solve for the unknown variables for any **two** components

Solving for s and t in the first two components	
2①+② $\quad 11t = 22$, $t = 2$	-5① +② $\quad 11s = -33$, $s = -3$

- Test the solution with the third component

sub $t = 2$ and $s = -3$ into ③

$$\begin{aligned} LS &= 3(2) + 3(-3) \\ &6-9 \\ &= -3 \\ &= RS \end{aligned}$$

∴ point of intersection: $\begin{aligned} x &= -2+5(2) = 8 \\ y &= 2 \\ z &= -3+3(2) = 3 \end{aligned}$

The point is (8,2,3) and the system is consistent and independent.

b) $\ell_1 : x + 2y = 3$ ① $\quad \ell_2 : 3x + 6y = 9$ ② → 3①-②: 0x+0y = 0

This is always true. ∴ lines are equal and have ∞'ly many points of intersection.

The system is consistent and dependent.

For solution, set one coordinate equal to a variable and solve for the other coordinate in terms of this variable.

Let $y = k$, sub into ①

$$x + 2k = 3$$
$$x = 3 - 2k$$

∴ The solution set is: $(3 - 2k, k) \qquad k \in \Re$

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c) $\ell_1 : \vec{r} = (2,-1,0) + t(1,2,-3)$

$\ell_2 : \dfrac{x+1}{-2} = \dfrac{y-1}{1} = \dfrac{z-2}{1}$

In parametric form: $\ell_1 :$ $x = 2 + t$, $y = -1 + 2t$, $z = -3t$ sub ℓ_1 into ℓ_2 $\dfrac{x+1}{-2} = \dfrac{y-1}{1} = \dfrac{z-2}{1}$

$$\frac{2+t+1}{-2} = \frac{-1+2t-1}{1} \qquad\qquad \frac{-1+2t-1}{1} = \frac{-3t-2}{1}$$

$$t = \frac{1}{5} \qquad\qquad t = 0$$

$t's$ are not equal ∴ no solution

$$\overrightarrow{d_1} = (1,2,-3) \qquad \overrightarrow{d_2} = (-2,1,1)$$

$$\overrightarrow{d_1} \neq m\overrightarrow{d_2}$$

∴ skew lines

Intersection of Lines Homework:

1. Find the intersection of the following lines. Classify the system as consistent or inconsistent, if possible dependent or independent

a) $\ell_1 : 5x - 2y + 25 = 0$
$\ell_2 : 5x - 2y - 5 = 0$

b) $\ell_1 : 2x - y + 14 = 0$
$\ell_2 : 2x - y + 3 = 0$

c) $\ell_1 : (x, y) = (-1,3) + t(-2,4)$
$\ell_2 : 2x + y - 1 = 0$

d) $\ell_1 : (x, y, z) = (-1,3,5) + t(1,-2,6)$ and $\ell_2 : \begin{cases} x = -13 + 3k \\ y = -8 + k \\ z = -1 + 3k \end{cases}$

e) $\ell_1 : (x, y) = (-1,4) + t(-1,4)$
$\ell_2 : 4x + y - 3 = 0$

f) $\ell_1 : (x, y, z) = (-1,3,5) + t(1,-2,6)$ and $\ell_2 : \begin{cases} x = -13 + 3k \\ y = -8 + k \\ z = 8 + 3k \end{cases}$

g) $\ell_1 : 2x + 3y - 30 = 0$
$\ell_2 : x - 2y + 13 = 0$

h) $\ell_1 : \begin{cases} x = 18 + 3k \\ y = -2 - 2k \end{cases}$ and $\ell_2 : \begin{cases} x = -5 + 2t \\ y = 4 + t \end{cases}$

i) $\ell_1 : \begin{cases} x = -1 + 3k \\ y = 1 + 4k \\ z = -2k \end{cases}$ and $\ell_2 : \begin{cases} x = -1 + 2t \\ y = 3t \\ z = -7 + t \end{cases}$

j) $\ell_1 : (x, y, z) = (2,1,0) + t(1,-1,1)$ and
$\ell_2 : (x, y, z) = (3,0,-1) + k(2,3,-1)$

Answers:

a) system is inconsistent, lines are parallel
b) system is inconsistent, lines are parallel
c) system is consistent and dependent, solution is $(0.5 - 0.5a, a), a \in \Re$
d) system is inconsistent, lines are skewed
e) system is inconsistent, lines are parallel
f) system is consistent and independent, solution is (2, -3, 23)
g) system is consistent and independent, solution is (3,8)
h) system is consistent and independent, solution is (3,8)
i) system is consistent and independent, solution is (5, 9, -4)
j) system is inconsistent, lines are skewed