Activity 7 – Day 1: Differentiation Rules Date:_______________

Part A – The Product Rule

The product rule is used for composite functions of the form $y = f(x)g(x)$ and is defined as follows:

The Product Rule

$$y = f(x)g(x)$$

then

$$\frac{dy}{dx} = \frac{d}{dx}\left[f(x) \bullet g(x)\right] = f(x) \bullet \frac{d}{dx} g(x) + \frac{d}{dx} f(x) \bullet g(x)$$

or

$$y' = f(x)g'(x) + f'(x)g(x)$$

Your first task is to prove the product rule from first principles.

Given: $F(x) = f(x)g(x)$ **Required to Prove**: $F'(x) = f'(x)g(x) + f(x)g'(x)$

Proof: $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{(x+h) - x}$ **Reminder**: $(x, F(x)) = (x, f(x)g(x))$

$(x+h, F(x+h)) = (x+h, f(x+h)g(x+h))$

$$F'(x) = \lim_{h \to 0} \frac{\qquad - \qquad}{h}$$

Now, write 2 extra terms that sum to 0 in the numerator: $-f(x)g(x+h) + f(x)g(x+h)$

$$F'(x) = \lim_{h \to 0} \frac{\qquad - f(x)g(x+h) + f(x)g(x+h) - \qquad}{h}$$

Form 2 fractions by grouping the first 2 terms together and the last 2 terms together.

$$F'(x) = \lim_{h \to 0} \left[\frac{\qquad - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - \qquad}{h} \right]$$

Factor any common object from the numerators of both fractions.

$$F'(x) = \lim_{h \to 0} \left[\frac{\qquad}{h} + \frac{\qquad}{h} \right]$$

Now, separate the large limit into 2 smaller limits.

$$F'(x) = \lim_{h \to 0} \left[\frac{\qquad}{h} \right] + \lim_{h \to 0} \left[\frac{\qquad}{h} \right]$$

Write each limit of products as products of limits

$$F'(x) = \lim_{h \to 0} (g(x+h)) \lim_{h \to 0} \left[\frac{\qquad}{h} \right] + \lim_{h \to 0} (f(x)) \lim_{h \to 0} \left[\frac{\qquad}{h} \right]$$

Therefore, $F'(x) =$ ____.

Part B – The Quotient Rule

The quotient rule is used for composite functions of the form $y=\frac{f(x)}{g(x)}$ and is defined as follows:

The Quotient Rule

$y=\frac{f(x)}{g(x)}$ then

$$\frac{dy}{dx}=\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{\frac{d}{dx}f(x)\cdot g(x)-f(x)\cdot\frac{d}{dx}g(x)}{\left[g(x)\right]^2}$$

or

$$y'=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

Your first task is to prove the quotient rule from first principles.

Given: $F(x)=\frac{f(x)}{g(x)}$

Required to Prove: $F'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$

Proof: $F'(x)=\lim\limits_{h\to 0}\frac{1}{h}\left[F(x+h)-F(x)\right]$

Reminder: $(x,F(x))=\left(x,\frac{f(x)}{g(x)}\right)$

$(x+h,F(x+h))=\left(x+h,\frac{f(x+h)}{g(x+h)}\right)$

$$F'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{\qquad}{\qquad}-\frac{\qquad}{\qquad}\right]$$

Next, obtain a common denominator.

$$F'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{\qquad}{\qquad}-\frac{\qquad}{\qquad}\right]$$

Now, write 1 fraction.

$$F'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{\qquad-\qquad}{\qquad}\right]$$

Now, write 2 extra terms that sum to 0 in the numerator: $-f(x)g(x)+f(x)g(x)$.

$$F'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{\qquad - f(x)g(x) + f(x)g(x) - \qquad}{\qquad}\right]$$

In the numerator, factor a positive object from the first 2 terms & a negative object from the last 2 terms. (The last 2 terms can be written in reverse order now).

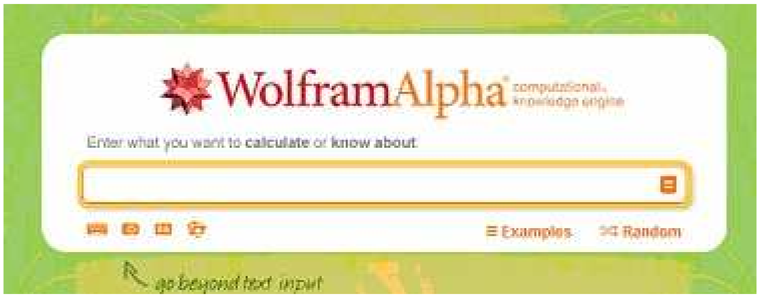
$$F'(x)=\lim_{h\to 0}\frac{1}{h}\left[\frac{(\qquad)[\qquad-\qquad]+(-\qquad)[\qquad-\qquad]}{\qquad}\right]$$

Multiply the 2 large numerator terms by the $\frac{1}{h}$ co-efficient.

$$F'(x)=\lim_{h\to 0}\left[\frac{(\qquad)\frac{[\qquad-\qquad]}{h}-(\qquad)\frac{[\qquad-\qquad]}{h}}{\qquad}\right]$$

Therefore, $F'(x)=$ ___________________________.

A valuable resource that can be used as you try these is the website: http://www.wolframalpha.com/
The website can be accessed through your computer, tablet or smartphone. If you are having difficulties, this website will guide you through any problem.



Its as simple as typing "Find derivative of x squared times sin of x". The website will recognize what you write and proved the step-by-step solution.

1. Find the derivative function for each of the following using the product rule

a) $f(x)=x(x^3-2x^2)$
b) $g(x)=(x^3-2x+1)(x^2-x)$
c) $G(y)=(y-\sqrt{y})\left(y+\dfrac{1}{\sqrt{y}}\right)$
d) $y=(x^2-3x)(4x+3)$
e) $f(x)=x^3e^x$
f) $y=x\sin x$
g) $f(x)=x\sin x+\cos x$
h) $h(x)=3^xe^x$
i) $g(x)=3^x(x^2-5x)$
j) $y=2^x\sin x$
k) $k(x)=3^x\ln x$
l) $f(x)=(3x-5)^2$
m) $\dfrac{d}{dx}(x^2\cos x)$

2. For $f(x)=\sqrt{x}(x^2-1)$, find the equation of the tangent and normal where $x=\dfrac{1}{4}$.

3. Find the equation of the normal to the curve $f(x)=(2-\sqrt{x})(1+\sqrt{x}+3x)$ when $x=1$.

4. Determine the equation of the tangent line to the graph of $y=xe^x$ at the point where $x=2$.

5. Find the equation of the tangent to the curve defined by $y=xe^x$ at the point $A(1,e)$.

6. Find all points at which the tangent to the curve defined by $y=x^2e^x$ is horizontal.

7. Find the equation of the tangent to $y=x^2\ln x$ at the point where $x=e$.

8. Determine the equation of the tangent line to the graph of $y=x\sin x$ at the point where $x=\dfrac{\pi}{2}$.

9. Find the equations of the tangent lines at the points where the graph of $y=(x-2)(3+x)$ intersects the x axis.

Answers:1) Go to Wolfram Alpha to find the derivatives

2) tangent line: $44x+64y+19=0$ normal: $512x-352y-293=0$

3) normal: $x+y-6=0$

4) tangent line: $3e^2x-y-4e^2=0$

5) tangent line: $2ex-y-e=0$

6) points where tangent line is horizontal: $(0,0)$ and $(-2,4e^{-2})$

7) tangent line: $3ex-y-2e^2=0$

8) tangent line: $y=x$

9) tangent lines: $5x-y-10=0$ and $5x+y+15=0$

1. Find the derivative function for each of the following using the quotient rule.

a) $f(t)=\dfrac{t^2+1}{t^2-1}$ b) $g(x)=\dfrac{x}{4+x^2}$ c) $y=\dfrac{1+2x}{4-x^2}$

d) $y=\dfrac{e^x}{x}$ e) $f(x)=\tan x=\dfrac{\sin x}{\cos x}$ f) $g(x)=\dfrac{x}{1-\cos x}$

g) $y=\dfrac{2^x}{\ln x}$ h) $h(x)=\dfrac{\cos x}{x}$ i) $y=\dfrac{\ln(x^2)}{x^2+1}$

j) $y=\dfrac{\sqrt[3]{x}}{e^x}$

2. Find the point of contact to $f(x)=\dfrac{x^3}{x+1}$ of each tangent with slope equal to 0.

3. The tangents to $f(x)=\dfrac{kx+4}{x-3}$ and $g(x)=\dfrac{3k}{2x-5}$ are parallel when $x=4$. Find the value of *k*.

4. Find the point(s) of contact to $y=\dfrac{x-4}{x-1}$ of all tangents which pass through the point $P(0,-8)$.

5. Determine the equation of the tangent to $h(x)=\dfrac{x^2-1}{3x}$ at $x=2$.

6. Determine the equation of the tangent to $h(x)=\dfrac{2e^x}{1+e^x}$ at the point $F(0,1)$.

7. Determine the equation of the tangent to $h(x)=\dfrac{2\ln x}{3x}$ at the point where $x=1$.

8. Find the slope of the tangent to $y=\dfrac{\sin x}{x}$ at $x=\dfrac{\pi}{4}$.

Answers:

1) Refer to wolfram alpha	2) (0,0) and $\left(\dfrac{-3}{2},\dfrac{27}{4}\right)$
3) $k=\dfrac{-12}{7}$	4) (2,-2) and $\left(\dfrac{2}{3},10\right)$
5) $5x-12y-4=0$	6) $x-2y+2=0$
7) $2x-3y-2=0$	8) $m=\dfrac{2\sqrt{2}}{\pi}-\dfrac{8\sqrt{2}}{\pi^2}$