For $\lim_{x \to a} f(x)$ to exist, $f(x)$ must approach the same value as x approaches a from the left, denoted $\lim_{x \to a^-} f(x)$, and as x approaches a from the right, denoted $\lim_{x \to a^+} f(x)$.

More precisely, for $\lim_{x \to a} f(x)$ to exist, the following three conditions must be met:

- $\lim_{x \to a^-} f(x)$ must exist,
- $\lim_{x \to a^+} f(x)$ must exist, and
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Example 1

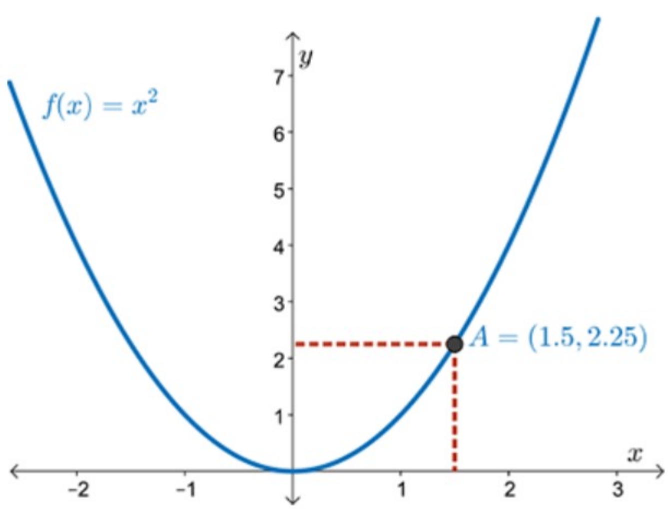
Left-Hand Limit

The value that $f(x)$ approaches as x moves along the graph from the left side is the **left-hand limit**.

Consider the graph of $f(x) = x^2$.

To find the left-hand limit of $f(x)$ as x approaches 2, denoted by $\lim_{x \to 2^-} f(x)$, we begin at a point on the parabola just left of $x = 2$.

Let's begin at $x = 1.5$.



x	y
1.50	2.25
1.58	2.49
1.64	2.75
1.72	3.03
1.80	3.31
1.88	3.61
1.96	3.92
2.00	4.00

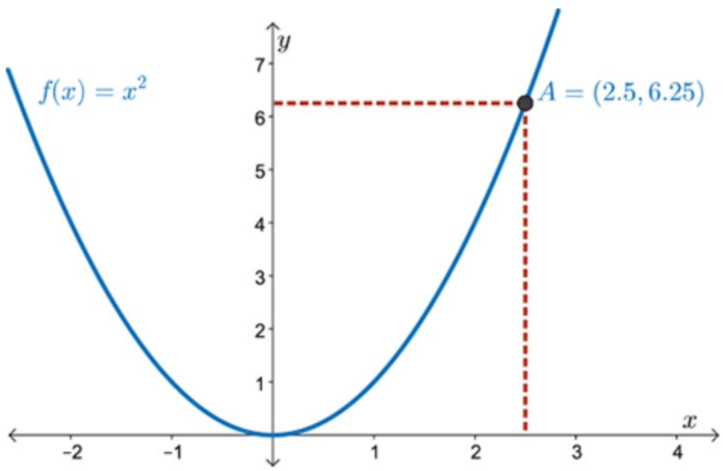
As we move point A from $1.5, 2.25$ towards an x value that is 2, we see that $f(x)$ approaches the value 4. This is denoted by $\lim_{x \to 2^-} f(x) = 4$.

Right-Hand Limit

The value that $f(x)$ approaches as x moves along the graph from the right side is the **right-hand limit**.

To find the right-hand limit of $f(x)$ as x approaches 2, denoted by $\lim_{x \to 2^+} f(x)$, we begin at a point on the parabola just right of $x = 2$.

Let's begin at $x = 2.5$. When x is 2.5, $f(x)$ is 6.25.



x	y
2.50	6.25
2.42	5.85
2.34	5.47
2.26	5.11
2.18	4.75
2.10	4.41
2.02	4.08
2.00	4.00

As x moves along the graph from the right side, we see that $f(x)$ approaches the value 4.

Evaluating the Limit From Both Sides

In conclusion, we're going to now consider the value that we are approaching from both sides. We have seen that the limit exists as we approach the value $x = 2$ for $f(x)$ from the left side and the right side. The limit exists because we were able to go to a point nearby to $x = 2$ on the left side and the right side, and we were able to approach the value when $x = 2$ from both sides.

Also, as x moves along the graph from the left and right sides of 2, we see that $f(x)$ approaches the value 4 from both directions.

Therefore, the limit of $f(x)$ from the left side is equal to the limit of $f(x)$ from the right side.

In summary, the following three conditions have been met:

$\lim_{x \to 2^-} f(x)$ exists and equals 4

$\lim_{x \to 2^+} f(x)$ exists and equals 4

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 4$

Thus, $\lim_{x \to 2} f(x)$ exists and equals 4.

Since $f(x)$ approaches the value 4 from both directions, $\lim_{x \to 2} f(x) = 4$

Example 2: Limits of Functions Having Discontinuities

Some functions have discontinuities.

Let's consider the rational function $f(x) = \dfrac{(x+1)(x-4)(x-5)}{2(x-5)}$.

This function has a removable discontinuity when $x = 5$. We know that since it has an equal factor $x - 5$ in the numerator and the denominator of this rational function.

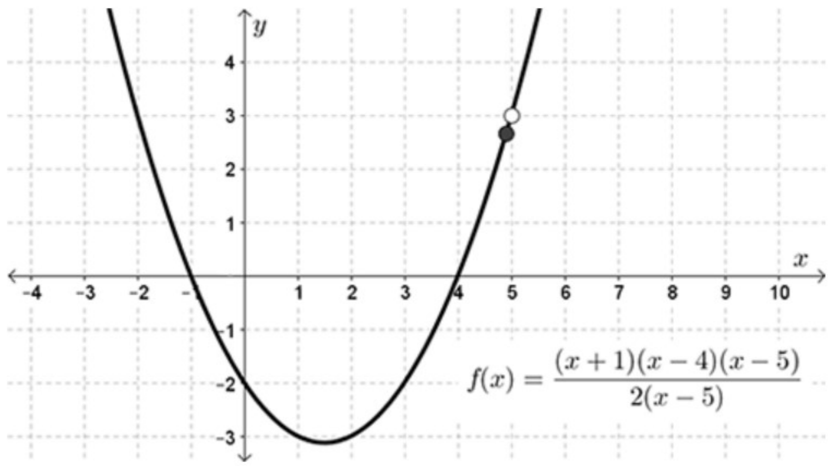
Since f is defined for all $x \neq 5$, there is a “hole” in the graph at $x = 5$ because this discontinuity is removable.

Now, let's consider the left side limit. We can approach the value $x = 5$ along the curve from the left side. As x approaches 5 from the left side, the limit of $f(x)$ approaches 3.

$$\lim_{x \to 5^-} f(x) = 3$$

$\uparrow$

as x appproaches
5 from the left.



We can also approach $x = 5$ along the curve from the right side. The limit of $f(x)$ as x approaches 5 from the right side is also 3.

$$\lim_{x \to 5^+} f(x) = 3$$

$\uparrow$

as x appproaches
5 from the right

In summary, the following three conditions have been met:

$\lim\limits_{x \to 5^-} f(x)$ exists and equals 3

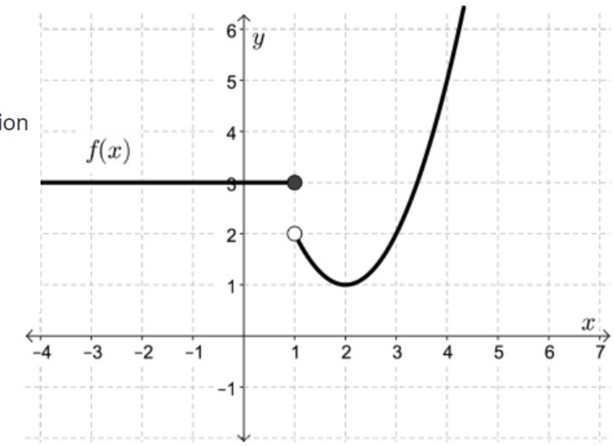
$\lim\limits_{x \to 5^+} f(x)$ exists and equals 3

$\therefore \lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^+} f(x) = 3$

Thus, $\lim\limits_{x \to 5} f(x)$ exists and equals 3.

Example 3: Limits of Piecewise Functions

A function may also have a jump discontinuity. Let's consider the piecewise function



$$f(x) = \begin{cases} 3 & \text{for } x \leq 1 \\ (x-2)^2 + 1 & \text{for } x > 1 \end{cases}$$

that has a jump discontinuity at $x = 1$.

The limit of $f(x)$ as x approaches 1 from the left side is 3.

$$\lim_{x \to 1^-} f(x) = 3$$

The limit of $f(x)$ as x approaches 1 from the right side is 2.

$$\lim_{x \to 1^+} f(x) = 2$$

In summary,

$\lim_{x \to 1^-} f(x)$ exists and equals 3

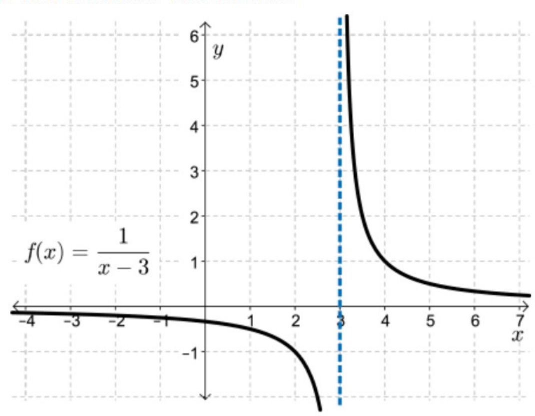
$\lim_{x \to 1^+} f(x)$ exists and equals 2

$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

Since $f(x)$ does not approach the same value in both directions as x approaches 1, this limit does not exist.

Example 4: Limits of Rational Functions

The last type of discontinuity that we will look at is a vertical asymptote. Let's consider the rational function $f(x) = \dfrac{1}{x-3}$, which has a vertical asymptote of $x = 3$.



In summary,the limit as x approaches 3 from the left side of $f(x)$ does not exist since as x approaches 3 from the left side, $f(x)$ does not approach a particular value. It approaches, instead, negative infinity.

$\lim_{x \to 3^-} f(x)$ does not exist

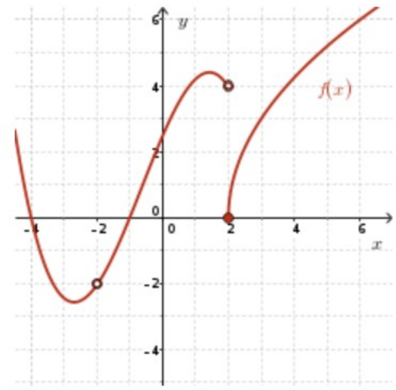
Also, the limit as x approaches 3 from the right side does not exist since $f(x)$ approaches positive infinity.

$\lim_{x \to 3^+} f(x)$ does not exist

Excercises

Solutions available at: https://courseware.cemc.uwaterloo.ca/11/assignments/62/12

1. Given the graph of $f(x)$, evaluate the following expressions involving $f(x)$.



a. $\lim_{x\to -1} f(x)$

b. $\lim_{x\to -2} f(x)$

c. $\lim_{x\to 2^+} f(x)$

d. $\lim_{x\to -2^-} f(x)$

e. $f(-2)$

f. $\lim_{x\to 2} f(x)$

g. $\lim_{x\to -2^+} f(x)$

h. $\lim_{x\to 2^-} f(x)$

i. $f(2)$

2. Graph the following piecewise function, then evaluate each limit below.

$$f(x) = \begin{cases} -x+4 & \text{if } x \le 3 \\ x-5 & \text{if } x > 3 \end{cases}$$

a. $\lim_{x\to 3^-} f(x)$

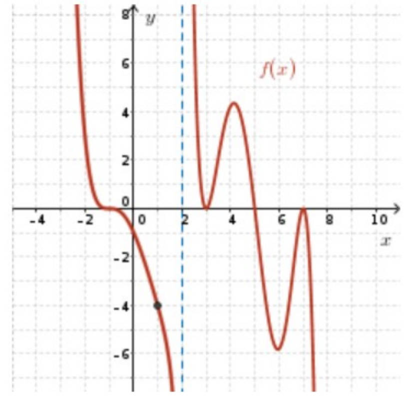
b. $\lim_{x\to 3^+} f(x)$

c. $\lim_{x\to 3} f(x)$

3. Sketch the graph of a function that has the following characteristics:

- $\lim_{x\to -1^-} f(x) = 3$
- $\lim_{x\to -1^+} f(x) = 1$
- $\lim_{x\to 3} f(x) = 2$
- $f(3)$ does not exist

4. Evaluate the following limits, given the graph.



a. $\lim_{x\to 1} f(x)$

b. $\lim_{x\to 3^-} f(x)$

c. $\lim_{x\to 3^+} f(x)$

d. $\lim_{x\to 2^-} f(x)$

e. $\lim_{x\to 2^+} f(x)$

f. $\lim_{x\to 2} f(x)$

5. Sketch the graph of a function that has the following characteristics:

- $\lim_{x\to 3^-} f(x) \to +\infty$
- $\lim_{x\to -1^+} f(x) \to -\infty$
- $\lim_{x\to 1} f(x) = 1$
- $f(0) = 0$

6. Consider the piecewise function $f(x)$ defined below, where A is a constant.

$$f(x) = \begin{cases} A^2x - 4A & \text{if } x \geq 2 \\ -2 & \text{if } x < 2 \end{cases}$$

Determine all values of A so that $\lim_{x\to 2} f(x)$ exists.

7. Consider the following piecewise function $f(x)$, where A and B are constants.

$$f(x) = \begin{cases} Ax + B & \text{if } x < -2 \\ x^2 + 2Ax - B & \text{if } -2 \leq x < 1 \\ 4 & \text{if } x > 1 \end{cases}$$

Determine all values of the constants A and B so that $\lim_{x\to -2} f(x)$ and $\lim_{x\to 1} f(x)$ both exist.