Applications of Cross Product and Dot Product – Part 1

Recall:

$\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta \qquad \vec{u}\cdot\vec{v}=(u_1,u_2)\cdot(v_1,v_2)=u_1v_1+u_2v_2$

$$\vec{u}\cdot\vec{v}=(u_1,u_2,u_3)\cdot(v_1,v_2,v_3)=u_1v_1+u_2v_2+u_3v_3$$

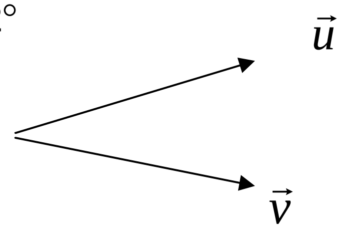
$\vec{u}\times\vec{v}=|\vec{u}||\vec{v}|\sin\theta\,\hat{e}$

$$\vec{u}\times\vec{v}=(u_1,u_2,u_3)\times(v_1,v_2,v_3)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=(u_2v_3-v_2u_3,v_1u_3-u_1v_3,u_1v_2-v_1u_2)$$

Applications:

 # 1 The sum or difference of two vectors cross product with the sum or difference of the same two vectors:

Ex. Find $(3\vec{u}-\vec{v})\times(\vec{u}-2\vec{v})$ where $|\vec{u}|=10, |\vec{v}|=17$ and $\theta=32°$



Ex. Find $(2\vec{x}-\vec{y})\times(5\vec{x}+\vec{y})$ where $\vec{y}=(2,2,-1), \quad \vec{x}=(-1,4,-3)$

#2 The triple scalar product: $(\vec{u}\times\vec{v})\cdot\vec{w}$

- If the triple scalar product equals zero then $\vec{u}, \vec{v}$ and $\vec{w}$ are coplanar.
- If the triple scalar product does not equal zero then $\vec{u}, \vec{v}$ and $\vec{w}$ are not coplanar.

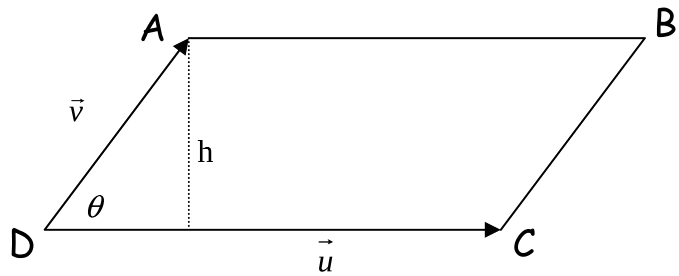
Ex. Given $\vec{a}=(2,1,0), \vec{b}=(-1,0,3)$ and $\vec{c}=(4,-1,1)$, calculate $(\vec{a}\times\vec{b})\cdot\vec{c}$

#3 Two vectors are collinear if they lie on the same line when placed tail to tail. (use scalar multiples)
Three vectors are coplanar if they lie on the same plane when placed tail to tail. (use triple scalar product)

Ex. Discuss the following sets of vectors wrt collinearity and coplanarity:
a) $\vec{a} = (2,1,0), \vec{b} = (-1,0,3)$ and $\vec{c} = (-4,-2,0)$
b) $\vec{a} = (2,1,0), \vec{b} = (-1,0,3)$ and $\vec{c} = (4,-1,1)$
c) $\vec{a} = (1,3,-1), \vec{b} = (2,1,1)$ and $\vec{c} = (-4,3,-5)$

 # 4 The Area of a Parallelogram

$$AREA = base \times height$$
$$= |\vec{u}|(h)$$
$$= |\vec{u}||\vec{v}|\sin\theta$$
$$= |\vec{u} \times \vec{v}|$$



Therefore, the area of a parallelogram is equal to the magnitude of the cross product of the two vectors which make up the parallelogram.

Note: area of triangle ADC= $\frac{1}{2}|\vec{u} \times \vec{v}|$

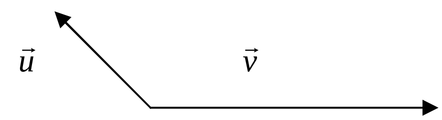
Ex. Given W(-1,5,2), X(6,-2,3) and Y(-2,-7,-3), find:

a) the co-ordinates of the point Z if WXYZ is a parallelogram.
b) the area of triangle WXY.
c) the area of parallelogram WXYZ.

Ex. If $|\vec{u}| = 10, |\vec{v}| = 17$ and $\theta = 32°$, find the area of the parallelogram that $\vec{u}$ and $\vec{v}$ make.

Homework for cross product applications

1. If $|\vec{u}| = 10$, $|\vec{v}| = 17$ and $\theta = 108°$, find $(3\vec{u} - \vec{v}) \times (\vec{u} + 5\vec{v})$



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2. $\vec{x} = 3\hat{i} - 2\hat{j} + 5\hat{k}$, $\vec{y} = (-1,4,3)$, find $(2\vec{x} - \vec{y}) \times (5\vec{x} + \vec{y})$ {(-182, -98, 70)}

3. Discuss the following sets of vectors with respect to collinearity and coplanarity:
   a) $\vec{a} = (3,5,6)$, $\vec{b} = (6,10,12)$ and $\vec{c} = (-3,-5,6)$ {$\vec{a}$ and $\vec{b}$ collinear, all three coplanar}
   b) $\vec{a} = (7,8,9)$, $\vec{b} = (-1,2,3)$ and $\vec{c} = (4,-1,1)$ {non-collinear, non-coplanar}
   c) $\vec{a} = (1,1,2)$, $\vec{b} = (1,2,0)$ and $\vec{c} = (3,-1,14)$ {coplanar}

4. Given W(-1,4,2), X(6,-2,5) and Y(-2,-7,1), find the area of triangle WXY. $\{45.9\ u^2\}$

5. Given three vertices of parallelogram WXYZ, W(-1,4,-2), X(6,-2,3) and Y(1,-6,-3), find the area of the parallelogram. $\{82.4\ u^2\}$

6. Find the unit vector perpendicular to both (2, 3, 0) and (0, 3, 4). $\left\{\left(\frac{6}{\sqrt{61}}, -\frac{4}{\sqrt{61}}, \frac{3}{\sqrt{61}}\right)\right\}$