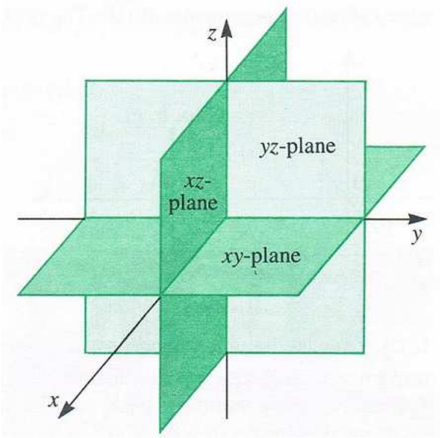
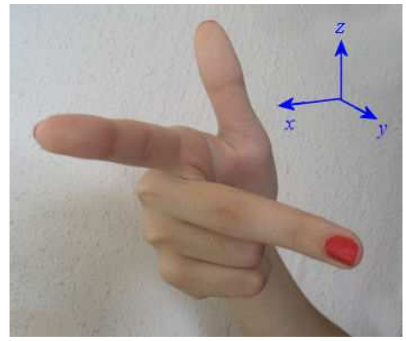
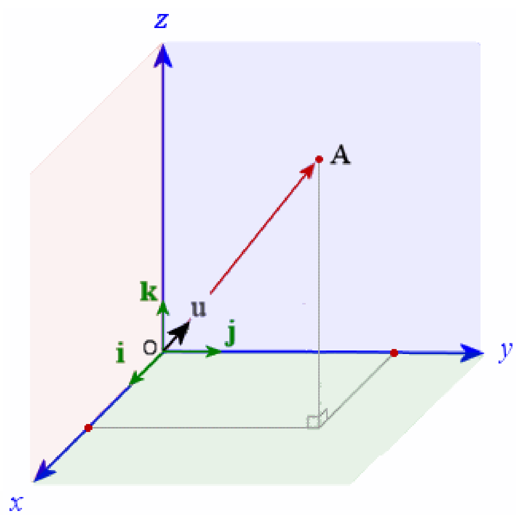
3-D Vectors

We can expand our 2-dimensional (*x*-*y*) coordinate system, R^2, into a **3-dimensional coordinate system**, R^3, using *x*-, *y*-, and *z*-axes.

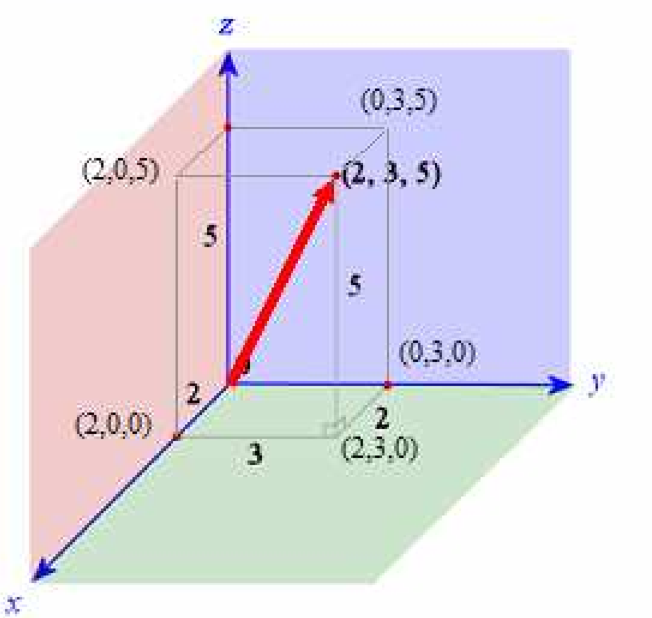






If we have $A(a,b,c)$, then

$$\overrightarrow{OA} = (a,b,c) = a\hat{i} + b\hat{j} + c\hat{k}$$

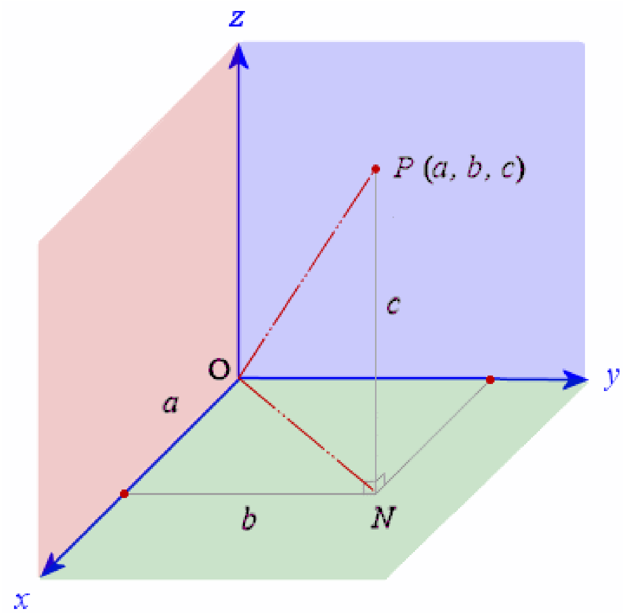


In 3-dimensional space, the point (2, 3, 5) is graphed as follows:

To reach the point (2, 3, 5), we move 2 units along the *x*-axis, then 3 units in the *y*-direction, and then up 5 units in the *z*-direction. This creates a prism.

Ex: Sketch $\vec{u} = (-1, 3, -5)$

Length of a Vector



The general point *P* (*a*, *b*, *c*) is shown on the 3D graph. The point *N* is directly below *P* on the *x*-*y* plane.

From Pythagoras' Theorem, we have

$$\left|\overrightarrow{ON}\right|^2 = a^2 + b^2,$$

$$\left|\overrightarrow{NP}\right| = c$$

$$\left|\overrightarrow{OP}\right|^2 = \left|\overrightarrow{ON}\right|^2 + \left|\overrightarrow{NP}\right|^2$$

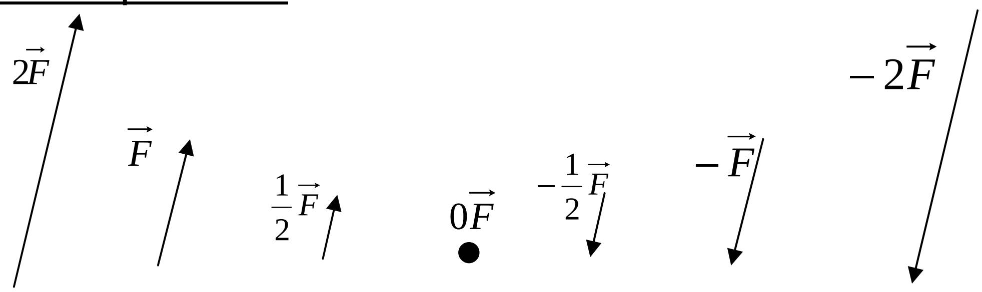
$$= a^2 + b^2 + c^2$$

∴ The distance from (0,0,0) to the point P(a,b,c) is given by $\left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2 + c^2}$

Example: Determine $\hat{u}$ if $\vec{u} = (-1, 3, -5)$

Operations with Geometric and Algebraic Vectors

Scalar Multiplication – Geometric



Notice that all vectors that are scalar multiples of some vector $\vec{F}$ are collinear.

If $\vec{v}$ is a vector and $k \in \Re$, then $k\vec{v}$ is also a vector such that

1) $|k\vec{v}| = |k||\vec{v}|$
2) the direction of $\vec{v}$ and $k\vec{v}$ are the same if $k > 0$ and opposite if $k < 0$.

We then have the following,

Two vectors $\vec{u}$ and $\vec{v}$ are collinear if and only if (iff) it is possible to find some nonzero scalar k such that $\vec{u} = k\vec{v}$.

Scalar Multiplication – Algebraic

If $\vec{u} = (a,b)$ then
$k\vec{u} = k(a,b) = (ka,kb)$

Proof:

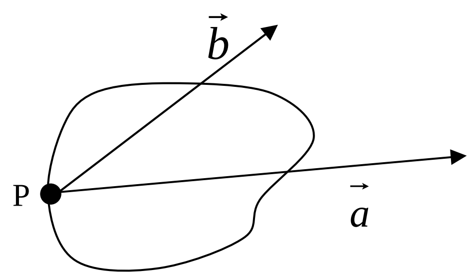
$$k(a,b) = k(a\hat{i} + b\hat{j})$$
$$= (ka\hat{i} + kb\hat{j})$$
$$= (ka, kb)$$

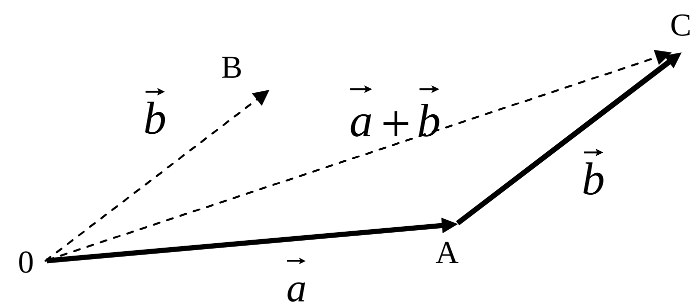
Vector Addition – Geometric

The resultant vector is the sum of two or more vectors; the combined effect of all vectors.

An equilibrant (force) is equal in magnitude but opposite in direction to the resultant.

Suppose we have two forces $\vec{a}$ and $\vec{b}$ applied to a point on a rock. The sum $\vec{a}+\vec{b}$ is the resultant.





From this we have the Triangle Law of Addition

If $\vec{a}$ and $\vec{b}$ are two vectors arranged "head to tail" the sum $\vec{a}+\vec{b}$ is represented by $\overrightarrow{OC}$.

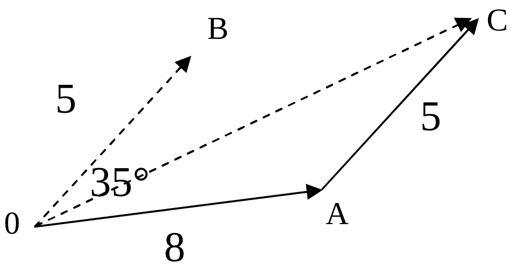
$$\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AC}$$
$$=\overrightarrow{OC}$$

Going from point O to A and then from A to C is the same as going from O to C.

NOTE: $|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}|$

Only equal if in same direction!
WHY?

Example: Find the magnitude and direction of the resultant of two vectors whose magnitudes are 5 and 8 respectively and the angle between them is 35°.



By the cosine law, we have

$|\overrightarrow{OC}| \approx 12.4 units$

By the sine law, we have

$\angle COA \approx 13.4°$

∴ The resultant vector $\overrightarrow{OC}$ has magnitude $|\overrightarrow{OC}| \approx 12.4 units$ and is 13.4° counter clockwise from $\overrightarrow{OA}$.

No N-E-W-S reference in this example

Vector Addition – Algebraic

Suppose we have vectors (a, b, c) and (d, e, f)

then (a, b, c) + (d, e, f) = (a + d, b + e, c + f)

Proof:

$$\begin{aligned}(a,b,c)+(d,e,f) &= a\hat{i}+b\hat{j}+c\hat{k}+d\hat{i}+e\hat{j}+f\hat{k}\\ &= (a+d)\hat{i}+(b+e)\hat{j}+(c+f)\hat{k}\\ &= (a+d, b+e, c+f)\end{aligned}$$

Example: If $\vec{u}=(3,4,-1)$ and $\vec{v}=(-1,2,4)$ and $\vec{w}=3\vec{u}+\vec{v}$ find:

a) $\hat{u}$ b) $-3\vec{u}$ c) $\vec{u}+\vec{v}$ d) $\vec{w}$

Homework on operations with vectors and 3-d Vectors 03:

1) a) Sketch $\vec{u} = (-1,2,-2)$. Find $|\vec{u}|$. Find $\hat{u}$. $\{|\vec{u}|=3,\ \hat{u} = \left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right)\}$

b) Sketch $\vec{u} = (4,-2,3)$. Find $|\vec{u}|$. Find $\hat{u}$. $\{|\vec{u}| = \sqrt{29},\ \hat{u} = \left(\frac{4}{\sqrt{29}}, \frac{-2}{\sqrt{29}}, \frac{3}{\sqrt{29}}\right)\}$

2) Find the magnitude of the following vectors:

a) (-12, -4, 6) {14}

b) (8, -27, 21) {35.1}

c) $\left(\frac{14}{27}, \frac{-22}{27}, \frac{-7}{27}\right)$ {1}

d) $\left(-\sqrt{2}, 2\sqrt{3}, \sqrt{2}\right)$ {4}

3) What single vector is equal to each of these sums?

a) $\overrightarrow{PT} + \overrightarrow{TS} + \overrightarrow{SQ}$ $\{\overrightarrow{PQ}\}$

b) $\overrightarrow{AC} - \overrightarrow{GE} + \overrightarrow{CE}$ $\{\overrightarrow{AG}\}$

c) $\overrightarrow{EA} - \overrightarrow{CB} + \overrightarrow{DB} + \overrightarrow{AD}$ $\{\overrightarrow{EC}\}$

d) $\overrightarrow{PT} - \overrightarrow{QT} + \overrightarrow{SR} - \overrightarrow{SQ}$ $\{\overrightarrow{PR}\}$

4) Find the sum of the vectors $\vec{u}$ and $\vec{v}$ if θ is the angle between them.

a) $|\vec{u}| = 12, |\vec{v}| = 21$ and $\theta = 70°$ $\{27.5,\ 24°$ to $\vec{v}\}$

b) $|\vec{u}| = 3, |\vec{v}| = 10$ and $\theta = 115°$ $\{9.1, 17.38°$ to $\vec{v}\}$

5) A tour boat travels 25 km due east and then 15 km $S50°E$. Represent these displacements in a vector diagram, then calculate the resultant displacement.

{37.74 km, $E14.8°S$}

6) Find a single vector equivalent to

a) $(2,-4)+(1,7)$ {(3, 3)}

b) $5(1,4)$ {(5, 20)}

c) $0(4,-5)$ {(0, 0)}

d) $(-6,0)+7(1,-1)$ {(1, -7)}

7) Simplify $(2\hat{i}+3\hat{j})+4(\hat{i}-\hat{j})$. $\{6\hat{i}-\hat{j}\}$

8) For the set of points, A(2, 0), B(3, 6), C(4, 1), and D(5, -5) use vectors to determine whether $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and whether $|\overrightarrow{AB}| = |\overrightarrow{CD}|$? {not parallel but magnitudes are equal}

9) Find x and y if $3(x,1)-2(2,y)=(2,1)$. $\{x=2,\ y=1\}$