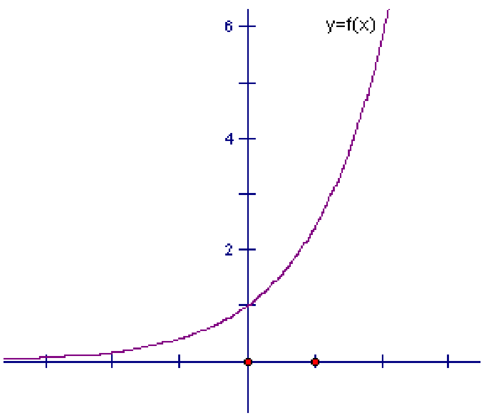
Investigating Derivatives, day 3 Date: ___________________

Recall Exponential Functions:

A function defined by an equation $f(x)=a^x$, where *a* is a real positive number, is called an *Exponential Function*.

There is a significant difference between the graphs and the properties of the function when $a>1$ and when $0<a<1$.

Case 1: $f(x)=a^x$, $a>1$. State the following:



Domain:

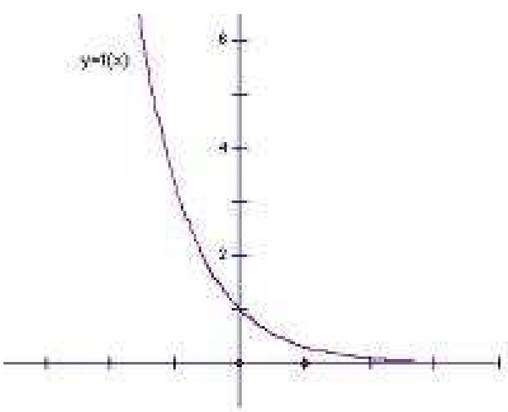
Range:

Asymptote:

The function is …………………… over the whole domain. I expect the derivative to always be …………………

For example:
$(2)^{-x}=\left(\frac{1}{2}\right)^x$

Case 2: $f(x)=a^x$, $0<a<1$. Remember this is equivalent to $f(x)=a^{-x}$, $a>1$.



Domain:

Range:

Asymptote:

The function is …………………… over the whole domain. I expect the derivative to always be …………………

Exploring the derivative of exponential functions

Use a graphing calculator to complete the following investigations.

For each of the functions below:

1) Sketch the function from its equation (use the default window: Zoom:Standard).

2) Complete the table of values for x, $f(x)$, and $f'(x)$ for at least three values of x (- 1, 0, and 1 are suggested but not mandatory choices). Round your answers to 3 decimal places.

3) Calculate the ratio $\frac{f'(x)}{f(x)}$.

4) Sketch the tangent line to the graph of $f(x)$ at *x = 0*. Determine the slope of the tangent line at $x=0$.

5) Summarize your findings.

Function and sketch

1) $f(x) = 2^x$

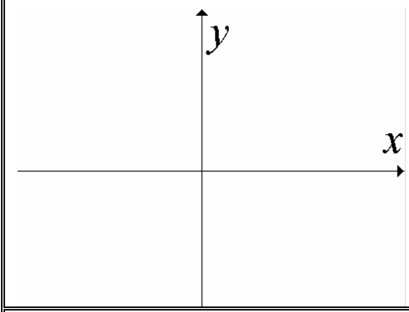


Table of values

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$

Slope of tangent line at $x = 0$

Conclusion:

Function and sketch

2) $f(x) = 3^x$

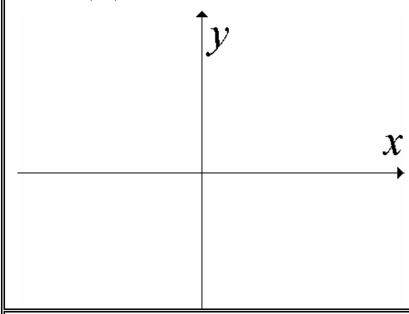


Table of values

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$

Slope of tangent line at $x = 0$

Conclusion:

Function and sketch

3) $f(x) = (0.2)^x$

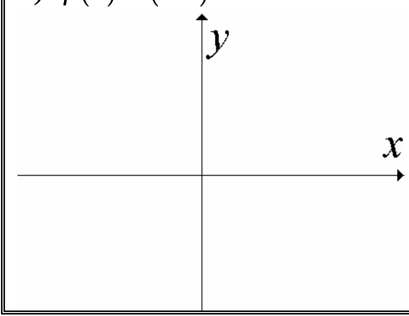


Table of values

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$

Slope of tangent line at $x = 0$

Conclusion:

Record what you noticed about each of the following:

1. The ratio $\frac{f'(x)}{f(x)}$

2. The relationship between $f(x)$, and $f'(x)$.

3. The slope of the tangent line at $x = 0$.

4. The derivative of the exponential function $f(x) = a^x$.

BIG IDEA - Did you notice that:

- ✓ The derivative of an exponential function is an exponential function that has been vertically stretched by a factor of $f'(0)$.

Observation: The slope of the tangent at $x=0$ for $y=2^x$ was less than one ($f'(0)<1$). The slope of the tangent at $x=0$ for $y=3^x$ was greater than one ($f'(0)>1$).

For each function, state the slope of the tangent at $x=0$.

1) $y=2^x$

$\left.\frac{dy}{dx}\right|_{x=0}=$ ____________

<1

2) $y=3^x$

$\left.\frac{dy}{dx}\right|_{x=0}=$ ____________

>1

Is there some function $y=b^x$ so that $\left.\frac{dy}{dx}\right|_{x=0}=1$?

Question: Does there exist a function $f(x)=b^x$, such that $f'(x)=b^x$. In other words, is there an exponential function that has $f(x)=f'(x)$? If possible, for what value of *b* will make it so?

For this to be true, we have

$$\begin{aligned} f'(x) &= f'(0)b^x \\ &= f'(0)f(x) \quad \text{and hence, } f'(0)=1. \\ &= (1)f(x) \end{aligned}$$

Also, consider this

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{b^{x+h}-b^x}{h} \\ &= \lim_{h\to 0} b^x \frac{b^h-1}{h} \\ &= b^x \lim_{h\to 0}\frac{b^h-1}{h} \end{aligned}$$

$$\therefore f'(0)=\lim_{h\to 0}\frac{b^h-1}{h}$$

Use trial and error for values for *b* and your graphing calculator to find $f(x)=b^x$ so that the slope of the tangent at the y-intercept is equal to 1 <u>or</u> when $f'(0)=\lim_{h\to 0}\frac{b^h-1}{h}=1$ b = ___________________

b	$\lim_{h\to 0}\frac{b^h-1}{h}$

Conclusions: If $f(x)=e^x$ then $\left.\frac{dy}{dx}\right\|_{x=0}=1$ and hence, $f'(x)=e^x$ where $e=$ ___________________

$f(x)=e^x$ is known as the natural exponential function

Extension: **Other occurrences of the irrational constant *e***

Use a graphing calculator to complete the following table:

Function	Sketch	Horizontal Asymptote	Limit
i) $y=\left(1+\frac{1}{x}\right)^x$			$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=$
ii) $y=(1+x)^{\frac{1}{x}}$			$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=$

iii) Evaluate at least 7 terms of the series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$ sum: ___________________

$n!=(n)(n-1)(n-2)\ldots(3)(2)(1)$

In general finding the derivative of $f(x)=b^x$:

Function $f(x)=b^x$	Derivative $f'(x)=f'(0)\cdot b^x$	$\ln(b)$	New Expression for $f'(x)$
$f(x)=(0.2)^x$			
$f(x)=(7)^x$			

So if:

$f(x)=b^x$

$g(x)=k(b^x)$

then:

$f'(x)=b^x\ln b$

$g'(x)=k(b^x)\ln b$

Examples:

1. Calculate the slope of the tangent to $f(x) = 2^x$ at $x = 12.5$

(4015.14)

2. For an exponential function $f(x)$ the ratio $\frac{f'(x)}{f(x)}$ = 1.946, determine the equation of the tangent line to the graph of the function at the point (0.5, 2.646).

($y = 5.149116x + 0.071442$)

<u>Practice Questions</u>:

1) Determine the equation of the tangent line of $f(x) = 4^x$ at $x = -3$.

2) Determine the equation of the tangent line of $f(x) = (0.5)^x$ at $x = 2$.

3) a) Determine the instantaneous rate of change of $g(x) = 6^x$ at $x = 5$, given that the slope of the tangent line at $x = 0$ is approximately 1.792.

   b) Determine the instantaneous rate of change of $y = -2(6^x)$ at $x = 5$.

4) The half-life of a radioactive material is about 1 year, thus the amount of material left after t years would be given by $A(t) = C(0.5)^t$ where C represents the original amount. Given that you have 1 g of the material initially, at what rate is the amount of material decreasing after 5 years? How would your answer change if the original amount of material was 20 g?

5) Through investigation you have found the rule for the derivative of an exponential function, $f(x) = a^x$ and the derivative of $f(x) = Ca^x$. Use this knowledge to determine the derivative from first principles of $f(x) = 2a^x$. Then extend this to determine the derivative from first principles of $f(x) = Ca^x$, where $C \in \Re$.

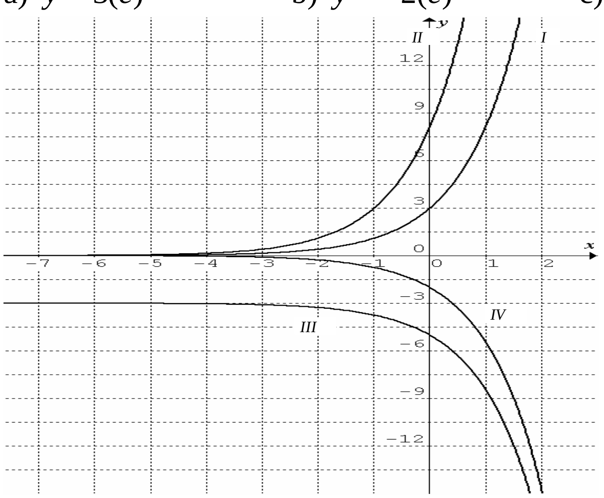
<u>Practice Questions</u>:

1) Evaluate (round to 4 decimal places)

   a) e^2 b) e^{-3} c) 2e + 3e d) 2^e

2) Match each function to the corresponding graph I, II, III, IV. Give (detailed) reasons for each choice.

   a) $y = 3(e)^{x+1}$ b) $y = -2(e)^x$ c) $y = 3(e)^x$ d) $y = -2(e)^x - 3$



Derivatives of Exponential Functions

Recall: finding the derivative of $f(x)=b^x$

Function $f(x)=b^x$	Derivative $f'(x)=f'(0)\cdot b^x$	$\ln(b)$: the slope of tangent of $f(x)=b^x$ when $x=0$	New Expression for $f'(x)$
$f(x)=(0.2)^x$			
$f(x)=(7)^x$			

Then we have:

If:

$f(x)=b^x$

$g(x)=k(b^x)$

then:

$f'(x)=b^x\ln b$

$g'(x)=k(b^x)\ln b$

where $f'(0)=\ln b$

Consider $f(x)=e^x$ **and its inverse**

1) A table of values is given for $f(x)=e^x$. On the grid, graph the following:

a. $f(x)=e^x$

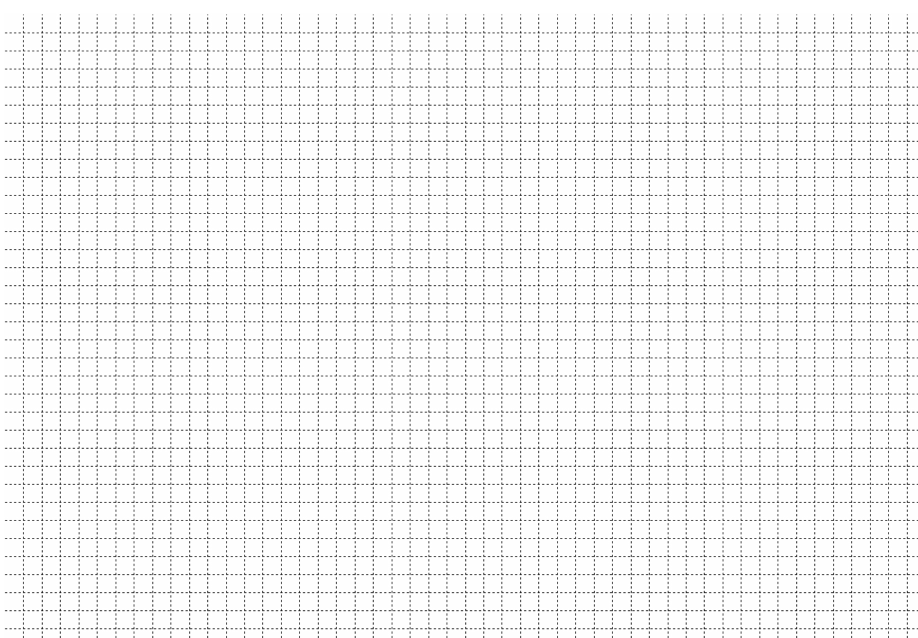
b. the line $y=x$

c. the reflection of the graph of $f(x)=e^x$ in the line $y=x$. Call it $g(x)$.

(Hint: For each point (*a*, *b*) on the graph of $f(x)$, the point (*b*, *a*) is its reflection image on the line $y=x$.)

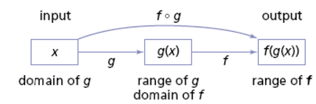
Choose appropriate scales for the axes.

x	**-2**	**-1**	**0**	**1**	**2**
e^x *(to the nearest hundredth)*	0.14	0.37	1	2.72	7.39



2) What is the inverse of the natural exponential function?

3) Since $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions, then $f(g(x)) = g(f(x)) = x$. This is called the Identity Property for inverse functions. Verifying the Identity properties using a graphing calculator



Input-output diagram of $f \circ g$.

This means that the y values of the function $g(x)$ become the x values of the function $f(x)$.

a) Enter the function $f(g(x)) = e^{\ln x}$ in the "Y=" editor. Graph it.

b) On what domain is $e^{\ln x}$ defined?

c) Enter the function $g(f(x)) = \ln(e^x)$ in the "Y=" editor. Graph it.

d) On what domain is $\ln(e^x)$ defined?

e) How do the functions $f(g(x)) = e^{\ln x}$ and $g(f(x)) = \ln(e^x)$ relate to the function $m(x) = x$?

(Hint: consider the domain)

<u>The derivative of the natural logarithmic function.</u>

Graph of the function $y = \ln x$ using your graphing calculator and complete the table below:

Value of x	Value of $y = \ln x$	Slope of the tangent at the x value.	Value of $\frac{1}{x}$
$\frac{1}{2}$			
3			
5			

Compare the value of the slope of the tangent with the value of $\frac{1}{x}$.

Put it together:

If $f(x) = \ln x$ then $f'(x) =$ __________

$g(x) = k(\ln x)$ $g'(x) =$ __________

Properties of the natural exponential function and the natural logarithmic function.

The natural exponential function $f(x)=e^x$ belongs to the family of exponential functions $f(x)=a^x$, where $a>1$. Similarly, the natural logarithmic function $g(x)=\ln x$ belongs to the broader family of logarithmic functions $g(x)=\log_a x$ where $a>1$. In fact,

1) $\ln x=\log_e x$
2) $\ln e=\log_e e=1$
3) $y=\ln x$ is the logarithmic form of $x=e^y$ the same way that $y=\log_a x$ is the logarithmic form of $x=a^y$
4) $y=\ln x$ is the inverse of $y=e^x$ the same way that $y=\log_a x$ is the inverse of $y=a^x$.

As a consequence, $e^{\ln x}=\ln e^x=x$, $x>0$

Recall: Laws of logarithms for any real numbers *a, b, x, y > 0* …These laws apply to natural logarithms too!

Log of a Product **Rule**	$\log_a(xy)=\log_a x+\log_a y$
Log of a Quotient **Rule**	$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$
Log of a Power **Rule**	$\log_a x^y=y\log_a x$

Ln *of a Product* Rule	$\ln(xy)=\ln x+\ln y$
Ln *of a Quotient* Rule	$\ln\left(\frac{x}{y}\right)=\ln x-\ln y$
Ln *of a Power* Rule	$\ln x^y=y\ln x$

Practice Exercises:

1. Use your calculator to evaluate each of the following expressions.
   a) $\ln 2$ b) $\ln 3$ c) $\ln 5$ d) $\ln 0.5$ e) $\ln 9$

2. Express each of the following as a single logarithm
   a) $\ln 5+\ln 8-\ln 4$ b) $\ln 81-\ln 3+\ln 2$ c) $2\ln x-\frac{1}{4}\ln y$

3. Evaluate $\ln\sqrt{e}$, without the use of technology.

4. Evaluate $\ln\left(\frac{1}{e}\right)+e^{\ln 0.2}-3\ln\sqrt[5]{e^3}$, without the use of technology.

5. State the domain of $y=\ln(x+2)$, then state the equation of the inverse.

6. State the domain of $y=\ln x^2$, then state the equation of the inverse.

7. Simplify each of the following
   a) $\ln\left(e^{-5\ln e}\right)$ b) $e^{-\ln(\ln e)}$ c) $\ln\left(e^{-2007\ \ln 1}\right)^{2008}$

8. Describe the transformations that $f(x)=\ln x$ has undergone to obtain each of the following functions, then state the equation of the inverse.
   a) $y=\ln(2x)+3$ b) $y=-\ln x-4$ c) $y=0.5\ln(x-1)+2$

9. What is the slope of the tangent to $y=\ln x$ for $x=7$?

10. Determine the equation of the tangent line of $f(x)=\ln x$ at *x = 2*.