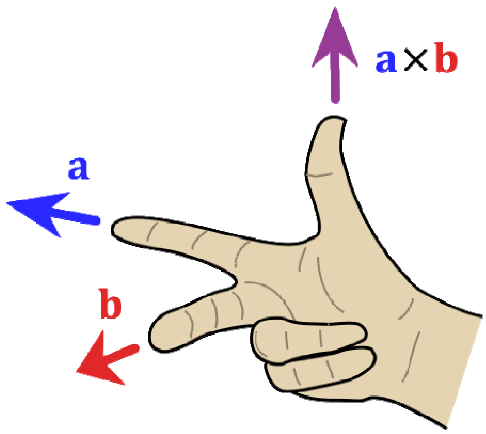
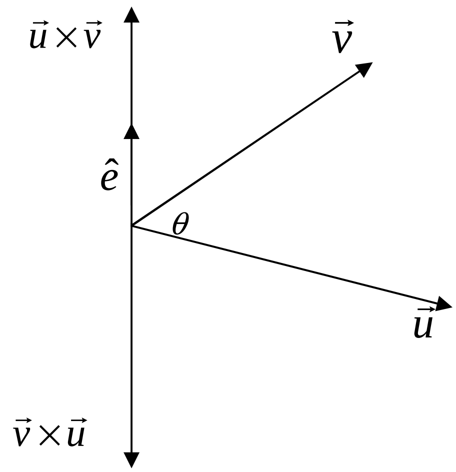
## The Cross Product

Definition: The cross product of two non-zero vectors $\vec{u}$ and $\vec{v}$ is $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\hat{e}$ where

$\theta$ is the angle between $\vec{u}$ and $\vec{v}$ tail to tail and
$\hat{e}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$ using a right handed system.

## Right handed system

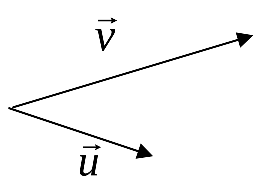



Note: the cross product only makes sense in three space.

Ex. Given: $|\vec{u}| = 5,\ |\vec{v}| = 8,\ \theta = 60°$

Find $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$




## Properties of the cross product

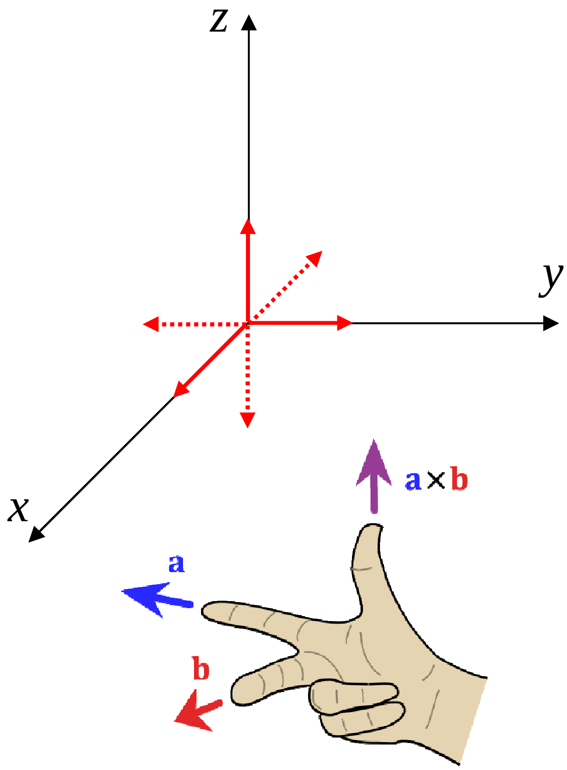
#1 $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$ not commutative

#2 $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ is distributive.

#3 $k(\vec{w}\times\vec{z}) = (k\vec{w})\times\vec{z} = \vec{w}\times(k\vec{z})$ scalar multiplication is associative.

Useful Facts:

If $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\hat{e}$ then




$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$$

$$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{i} = -\hat{k}$$

$$\hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{j} = -\hat{i}$$

$$\hat{k}\times\hat{i} = \hat{j} \qquad \hat{i}\times\hat{k} = -\hat{j}$$

The cross product in algebraic form (U1 A5 P2)

$$
\begin{aligned}
\vec{u}\times\vec{v} &= (u_1,u_2,u_3)\times(v_1,v_2,v_3) \\
&= (u_1\hat{i}+u_2\hat{j}+u_3\hat{k})\times(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) \\
&= u_1\hat{i}\times(v_1\hat{i}+v_2\hat{j}+v_3\hat{k})+u_2\hat{j}\times(v_1\hat{i}+v_2\hat{j}+v_3\hat{k})+u_3\hat{k}\times(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) \\
&= u_1v_1(\hat{i}\times\hat{i})+u_1v_2(\hat{i}\times\hat{j})+u_1v_3(\hat{i}\times\hat{k})+ \\
&\qquad u_2v_1(\hat{j}\times\hat{i})+u_2v_2(\hat{j}\times\hat{j})+u_2v_3(\hat{j}\times\hat{k})+ \\
&\qquad\quad u_3v_1(\hat{k}\times\hat{i})+u_3v_2(\hat{k}\times\hat{j})+u_3v_3(\hat{k}\times\hat{k}) \\
&= u_1v_2\hat{k}+u_1v_3(-\hat{j})+u_2v_1(-\hat{k})+u_2v_3\hat{i}+u_3v_1\hat{j}+u_3v_2(-\hat{i}) \\
&= (u_2v_3-u_3v_2)\hat{i}+(u_3v_1-u_1v_3)\hat{j}+(u_1v_2-u_2v_1)\hat{k} \\
&= (u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)
\end{aligned}
$$

$$\therefore \vec{u}\times\vec{v} = (u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$$

A second method called <u>down products minus up products</u>.

| | | |
|---|---|---|
| $u_1$ | $u_2 \quad u_3 \quad u_1 \quad u_2$ | $u_3$ |
| $v_1$ | $v_2 \quad v_3 \quad v_1 \quad v_2$ | $v_3$ |

write the first vector side by side twice

write the second vector side by side twice directly under the first one

Ignore the first and last column

$$u_2v_3 \qquad u_3v_1 \qquad u_1v_2$$

Calculate down products $\searrow$

$$v_2u_3 \qquad v_3u_1 \qquad v_1u_2$$

Calculate up products $\nearrow$

$$\left(u_2v_3 - v_2u_3,\ u_3v_1 - v_3u_1,\ u_1v_2 - v_1u_2\right)$$

Subtract down products – up products to get resulting cross product

A third method called <u>the determinant method</u>.

Create

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Ex. Given $\vec{a} = (-2,4,3), \vec{b} = (1,7,-2)$, find $\vec{a} \times \vec{b}$.

Show that $\vec{a} \times \vec{b} \cdot \vec{a} = 0$ and $\vec{a} \times \vec{b} \cdot \vec{b} = 0$.

**<u>Properties of the Cross Product</u>**

Verify the following properties of the cross product either geometrically or algebraically or both ways where appropriate.

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

$$a(\vec{u} \times \vec{v}) = (a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v}),\ a \in \Re$$

$$\vec{u} \times \left(\vec{v} + \vec{w}\right) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$\vec{u} \times \vec{u} = \vec{0}$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{k} = -\hat{j} \qquad \hat{k} \times \hat{i} = \hat{j}$$

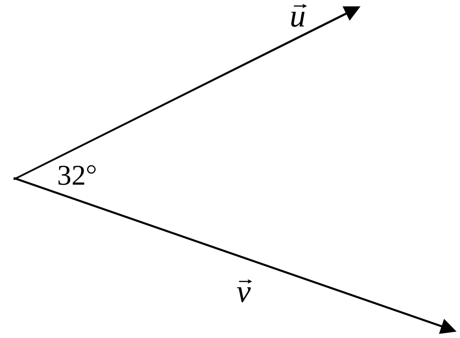
$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{k} \times \hat{j} = -\hat{i} \qquad \hat{j} \times \hat{k} = \hat{i}$$

## <u>Homework for Cross Product</u>

1. Given vectors $\vec{u}$ and $\vec{v}$ as shown, with $|\vec{u}| = 10$, $|\vec{v}| = 17$ find:

   $\vec{v} \times \vec{u}$ (include direction)

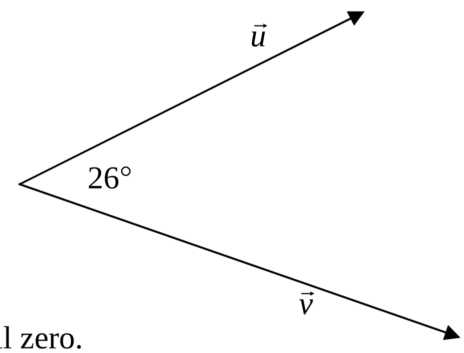



2. Given $\vec{y} = 2\hat{i} + 2\hat{j} - \hat{k}$, $\vec{z} = (3,-2,1)$, $A(-1,-3,5)$, $B(-3,-2,1)$ ,find $\vec{z} \times \vec{y}$, $\vec{z} \times \overrightarrow{AB}$

3. Given $\vec{y} = (1,5,-3)$, $\vec{z} = (3,-2,1)$ show that $\vec{z} \times \vec{y} = -(\vec{y} \times \vec{z})$

4. Given vectors $\vec{u}$ and $\vec{v}$ as shown, with $|\vec{u}| = 5$, $|\vec{v}| = 9$ find:

   $\vec{u} \times \vec{v}$ (include direction)




5. If $\vec{w} = \vec{u} \times \vec{v}$, explain why $\vec{w} \cdot \vec{u}$, $\vec{w} \cdot \vec{v}$ and $\vec{w} \cdot (a\vec{u} + b\vec{v})$ are all zero.

6. State whether each of the following expressions are vectors, scalars or meaningless.

| | | |
|---|---|---|
| a) $\vec{a} \cdot (\vec{b} \times \vec{c})$ | b) $(\vec{a} \cdot \vec{b}) \times (\vec{b} \cdot \vec{c})$ | c) $(\vec{a} + \vec{b}) \cdot \vec{c}$ |
| d) $\vec{a} \times (\vec{b} \cdot \vec{c})$ | e) $(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c})$ | f) $(\vec{a} + \vec{b}) \times \vec{c}$ |
| g) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$ | h) $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c})$ | i) $(\vec{a} \times \vec{b}) - \vec{c}$ |
| j) $\vec{a} \times (\vec{b} \times \vec{c})$ | k) $(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{c})$ | l) $(\vec{a} \cdot \vec{b}) - \vec{c}$ |

| | | | |
|---|---|---|---|
| 1.{ $90.9\hat{e}_{out}$ } | 2. {(0, 5, 10),(7, 10, -1)} | 4. {$19.7\,\hat{e}_{in}$ } | 6. {vectors f, h, i, j scalars a, c, e, k} |