Properties of the Dot Product

Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$

Prove: $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
Proof:

Algebraic

$$\begin{aligned}\vec{u}\cdot\vec{v} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= v_1u_1 + v_2u_2 + v_3u_3\\ &= \vec{v}\cdot\vec{u}\end{aligned}$$

Geometric

$$\begin{aligned}\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos\theta\\ &= |\vec{v}||\vec{u}|\cos\theta\\ &= \vec{v}\cdot\vec{u}\end{aligned}$$

Prove: $a(\vec{u}\cdot\vec{v}) = (a\vec{u})\cdot\vec{v} = \vec{u}\cdot(a\vec{v}),\ a\in\Re$
Proof:

Algebraic

$$\begin{aligned}a(\vec{u}\cdot\vec{v}) &= a(u_1v_1 + u_2v_2 + u_3v_3)\\ &= au_1v_1 + au_2v_2 + au_3v_3\\ &= (au_1, au_2, au_3)\cdot(v_1, v_2, v_3)\\ &= (a\vec{u})\cdot\vec{v}\end{aligned}$$

Geometric

$$\begin{aligned}a(\vec{u}\cdot\vec{v}) &= a|\vec{u}||\vec{v}|\cos\theta\\ &= |a\vec{u}||\vec{v}|\cos\theta\\ &= (a\vec{u})\cdot\vec{v}\end{aligned}$$

Prove: $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
Proof:

$$\begin{aligned}\vec{u}\cdot(\vec{v}+\vec{w}) &= (u_1, u_2, u_3)\cdot(v_1+w_1, v_2+w_2, v_3+w_3)\\ &= u_1\cdot(v_1+w_1) + u_2\cdot(v_2+w_2) + u_3\cdot(v_3+w_3)\\ &= (u_1v_1 + u_2v_2 + u_3v_3) + (u_1w_1 + u_2w_2 + u_3w_3)\\ &= \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}\end{aligned}$$

Prove: $\vec{u}\cdot\vec{u} = |\vec{u}|^2$

Proof:

Algebraic

$$\begin{aligned}\vec{u}\cdot\vec{u} &= (u_1, u_2, u_3)\cdot(u_1, u_2, u_3)\\ &= u_1^2 + u_2^2 + u_3^2\\ &= |\vec{u}|^2\end{aligned}$$

Geometric

The angle between $\vec{u}$ and itself is 0° and cos 0° = 1, so that

$$\begin{aligned}\vec{u}\cdot\vec{u} &= |\vec{u}||\vec{u}|\cos 0^\circ\\ &= |\vec{u}|^2\end{aligned}$$

Prove: $\vec{u}\cdot\vec{v} = 0$ iff $\vec{u}\perp\vec{v}$
Proof:
If $\vec{u}$ and $\vec{v}$ are perpendicular, then the angle between them is 90° and, since cos 90° = 0, we have $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos 90^\circ$.

Conversely, if $\vec{u}\cdot\vec{v} = 0$, then $|\vec{u}||\vec{v}|\cos\theta = 0$, where θ is the angle between the vectors.

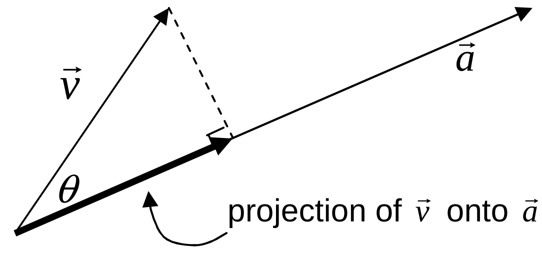
Since $\vec{u}$ and $\vec{v}$ are non-zero, we must have $\cos\theta = 0$ and therefore $\theta = \pm 90^\circ$.

More Applications of Dot Product - Projections

Finding the **projection** of one vector onto another vector.

To obtain a projection "drop" a vector perpendicular onto another vector.

Ex



projection of $\vec{v}$ onto $\vec{a}$ → $\mathrm{Proj}_{\vec{a}}\,\vec{v}$

This projection is a vector in the direction of $\vec{a}$ so it is a scalar multiple of $\vec{a}$. The magnitude of the projection, $|\mathrm{Proj}_{\vec{a}}\,\vec{v}|$, is given by $|\vec{v}|\cos\theta$

$$|\mathrm{Proj}_{\vec{a}}\,\vec{v}| = |\vec{v}|\cos\theta$$

← Recall from dot product: $\cos\theta = \dfrac{\vec{a}\bullet\vec{v}}{|\vec{a}||\vec{v}|}$

$$= |\vec{v}|\frac{\vec{a}\bullet\vec{v}}{|\vec{a}||\vec{v}|}$$

$$= \frac{\vec{a}\bullet\vec{v}}{|\vec{a}|}$$

"The size of the projection"

$$\therefore\ \mathrm{Proj}_{\vec{a}}\,\vec{v} = \frac{\vec{a}\bullet\vec{v}}{|\vec{a}|}\hat{a}$$

→ (in terms of $\hat{a}$)

Scalar multiple that is the size of the projection.

$\hat{a}$ is the unit vector in direction of $\vec{a}$

$$\mathrm{Proj}_{\vec{a}}\,\vec{v} = \frac{\vec{a}\bullet\vec{v}}{|\vec{a}|}\left(\frac{\vec{a}}{|\vec{a}|}\right)$$

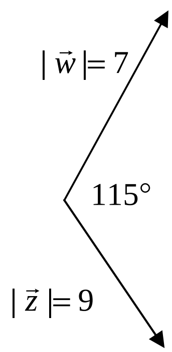
→ (in terms of $\vec{a}$)

$$= \frac{\vec{a}\bullet\vec{v}}{|\vec{a}|^2}\vec{a}$$

Recall: $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

Ex 1 Given $\vec{w} = (-3,5,2)$ $\vec{z} = (2,4,-3)$. Find $\text{Proj}_{\vec{z}}\vec{w}$ as an algebraic vector. $\left(\frac{16}{29}, \frac{32}{29}, \frac{-24}{29}\right)$

Ex 2 Find the $\text{Proj}_{\vec{z}}\, \vec{w}$.



Work

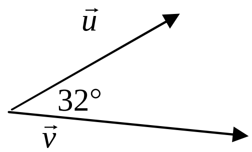
Ex. If a 15 N force, acting in the direction of $\vec{v} = (1,1,1)$, moves an object from P(-2, 1,-4) to Q(5, 6,-1), calculate the work done. The distance is in meters.

Homework on dot product and Projections

1. Given vectors $\vec{u}$ *and* $\vec{v}$ as shown, find

$\Pr oj\,(\vec{u}\ onto\ \vec{v})$ (in terms of $\hat{v}$)

{ $8.48\hat{v}$ }



$|\vec{u}| = 10,\ |\vec{v}| = 17$

2. Given $\vec{y} = 2\hat{i} + 2\hat{j},\quad \vec{x} = (-1,4),\quad \vec{z} = (3,-2)$
$A(-1,-3),\quad B(-3,-2)$

Find: $\Pr oj_{\overrightarrow{AB}}\,\vec{x}$ $\left(\frac{-12}{5}, \frac{6}{5}\right)$

3. Given $\vec{u} = (5,-4)$ *and* $\vec{v} = (-3,1)$, find the $\Pr oj(\vec{u}\ onto\ \vec{v})$ as an algebraic vector. $\{\left(\frac{57}{10}, \frac{-19}{10}\right)\}$

4. Find the work done if a force of 45 N in the direction of $\vec{r} = (4,-7)$ moves an object from G(1,6) to H(7,3). {251.1 Joules}

5. Given $\vec{y} = 2\hat{i} + 2\hat{j} + 7\hat{k},\quad \vec{x} = (-1,4,6),\quad \vec{z} = (3,-2,-3),\quad A(-1,-3,5),\quad B(-3,-2,4)$

Find $\Pr oj_{\overrightarrow{AB}}\,\vec{x}$ { $\vec{0}$ }

6. Find the work done by the force $\vec{f} = (-1,3,6)$, which moves an object from P(3, 1,-2) to Q(4, 5,1). { 29 Joules}

7. Find the work done if a force of 45 N in the direction of $\vec{r} = (4,-7,-2)$ moves an object from G(1,6,5) to H(7,3,7). {222.1 Joules }

8. Given $\vec{u} = (2,-1,6)$ *and* $\vec{v} = (1,3-4)$, find the $\Pr oj(\vec{v}\ onto\ \vec{u})$ as an algebraic vector. $\{\left(\frac{-50}{41}, \frac{25}{41}, \frac{-150}{41}\right)\}$

9. For $\vec{u} = (1,5,8)$ and $\vec{v} = (-1,3,-2)$, verify that:
   a) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
   b) $\vec{u}\cdot\vec{u} = |\vec{u}|^2$ and $\vec{v}\cdot\vec{v} = |\vec{v}|^2$
   c) $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = |\vec{u}|^2 - |\vec{v}|^2$
   d) $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = |\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2$
   e) $2(\vec{u}\cdot\vec{v}) = (2\vec{u})\cdot\vec{v} = \vec{u}\cdot(2\vec{v})$

10. If $\vec{u} = (2,2,-1),\ \vec{v} = (3,-1,0)$ and $\vec{w} = (1,7,8)$, verify that $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$